IFAC Workshop on Control Applications of Optimization, Haifa, Israel, Dec. 1995. To appear in published proceedings.

AN APPROACH TO SOLVING TWO TIME-SCALE TRAJECTORY OPTIMIZATION PROBLEMS

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Abstract. The indirect method of solving optimal control problems requires the solution of a Hamiltonian boundary-value problem (HBVP). Geometric properties of Hamiltonian systems and the notion of a dichotomy guide the development of a solution approach for HBVPs that involve two distinct time scales and boundary layers.

Key Words. Optimal Control; Dichotomy transformation; Hamiltonian Systems.

1. INTRODUCTION

The methods for solving optimal control problems are usually classified as direct or indirect. Indirect methods involve determining extremals by solving the Hamiltonian boundary-value problem posed by the first-order necessary conditions. Hamiltonian systems have the following property: the closure of an open set of initial conditions in the state-costate space propagated forward in time according to the Hamiltonian differential equations will maintain constant "volume". If the set is contracting in some directions, it is expanding in an equal number of other directions. Thus, any solution method that involves integration with initial conditions (or final conditions) that are in error will be plagued by error amplification in certain directions. The usual statement is that indirect methods require good initial estimates due to the high sensitivity. The sensitivity is especially problematic if the Hamiltonian dynamics evolve on two or more disparate time scales.

The solution approach studied in this paper utilizes a transformation to decouple the expanding and contracting behavior, so that the solution to the Hamiltonian boundary-value problem can be constructed by integrating the contracting part of the solution in forward time and the growing part of the solution in backward time. In this way, errors are not amplified and an iterative solution scheme can converge. The decoupling transformation is constructed from solutions to a differential Riccati equation. The use of decoupling transformations for linear boundary-value problems is well-known in the numerical analysis literature, see e.g. O'Malley and Anderson (1982) and Ascher *et al.* (1988). Our use of decoupling transformations for nonlinear Hamiltonian boundary-value problems is inspired by the Computational Singular Perturbation (CSP) method of Lam (1993, 1994) and its geometric interpretation by Mease (1995). Our work has also benefited from the ideas and results of Kokotovic and colleagues summarized in Kokotovic *et al.* (1986). Previous consideration of the application of the CSP method to optimal control problems can be found in Ardema (1990).

For the case when the Hamiltonian system, in the region of the state-costate space of interest, exhibits boundary-layer type, two time-scale behavior, the conceptual basis for an approximate numerical solution method is described. The method is motivated by the analytical method of matching asymptotic expansions for singularly perturbed boundary-value problems, but does not require the Hamiltonian system to be in the so-called *standard form* (see Kokotovic *et al.*, 1986). The solutions of a special class of two time-scale problems can be constructed from the solutions to two nonlinear infinite horizon regulator problems. An algorithm for solving such regulator problems is given. See also Rao and Mease (1995a,b) for the presentation of a variation of this algorithm and some numerical experience.

2. HAMILTONIAN BOUNDARY-VALUE PROBLEM

The optimal control problem to be considered is: Find the piecewise continuous control u that minimizes the scalar cost

$$J = \int_0^{t_f} L(x,u)dt \tag{1}$$

subject to the differential constraint on the state x and control

$$\dot{x} = f(x,u) \tag{2}$$

the initial condition $x(0) = x_0$, and the terminal condition $x(t_f) = x_f$. Let $x(t) \in R^n$ and $u(t) \in R^m$. The first-order necessary conditions for a weak or strong local minimum lead to a Hamiltonian boundary-value problem (HBVP) for the extremal trajectories. The HBVP is composed of the Hamiltonian differential system

$$\frac{d}{dt}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} H_\lambda \\ -H_x \end{pmatrix} \tag{3}$$

and the boundary conditions

$$x(0) = x_0 \tag{4}$$
$$x(t_f) = x_f \tag{5}$$

where $\lambda(t) \in R^n$ is the costate, taken here to be a row vector, and $H(x,\lambda) = L(x,u(x,\lambda)) + \lambda f(x,u(x,\lambda))$ is the optimal Hamiltonian. We assume that H is a smooth function of x and λ. The x-space and the (x,λ)-space are referred to here as the state space and the phase space, resp.

The focus of our attention is on the HBVP. Although the specifics of the approach to be discussed are given for the HBVP corresponding to the above form of optimal control problem, the general approach is applicable to HBVPs corresponding to other forms of optimal control problems as well.

3. SUPPORTING THEORY

Liouville's theorem (discussed e.g. in Guckenheimer and Holmes, 1983) states that the divergence of a vector field is zero if and only if the corresponding flow preserves volume. The right-hand-side of Eq. (3) is a vector field on the phase space in that it assigns a $2n$-dimensional vector to each point (x,λ) in the phase space. For a dynamical system of the general form

$$\dot{p}(t) = f(p(t)) \tag{6}$$

where $p(t) \in R^{2n}$, the divergence of the vector field is

$$div f = \sum_{i=1}^{2n} \frac{\partial f_i}{\partial p_i} \tag{7}$$

A consequence of the Hamiltonian form of vector field in Eq. (3) is that

$$div \begin{pmatrix} H_\lambda \\ -H_x \end{pmatrix}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\frac{\partial H}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_i}\frac{\partial H}{\partial x_i} \right) = 0 \tag{8}$$

Thus Liouville's theorem tells us that the flow corresponding to the vector field of Eq. (3) preserves volume. The terms flow and volume arise from an anology with fluid dynamics. Consider a set $S \subset R^{2n}$ defined as the closure of an open, connected set of initial phase points. A consequence of Liouville's theorem is that, if the flow contracts S in some directions, there will be an equal number of directions in which it expands S. This is the fundamental property of a Hamiltonian system that can make a HBVP difficult to solve numerically.

By considering the linearized motion about a reference trajectory, one can more precisely characterize the contracting and expanding behavior using the notion of a dichotomy (see Ascher *et al.* (1988) or Kokotovic *et al.* (1986)). For a dynamical system of the general form in Eq. (6), the variational equation takes the form

$$\delta\dot{p}(t) = \frac{\partial f}{\partial p}(p(t))\delta p(t) \tag{9}$$

Let Φ denote the transition matrix with initial condition $\Phi(0) = I$ that satisfies the variational equation along a particular trajectory $p(\cdot)$. The variational equation along this trajectory has an *exponential dichotomy* on an interval $[0,t_f]$, if there exist a constant projection matrix Π of rank $r, 0 \le r \le 2n$, a positive constant K of moderate size, and positive constants σ and μ, such that

$$\|\Phi(t)\Pi\Phi^{-1}(s)\| \le Ke^{-\sigma(t-s)}, \quad t \ge s \tag{10}$$
$$\|\Phi(t)(I-\Pi)\Phi^{-1}(s)\| \le Ke^{-\mu(s-t)}, \quad t \le s \tag{11}$$

"Moderate size" means that K should not be large enough to accommodate exponentially increasing motion in inequalities (10) and (11).

The variational equations corresponding to the Hamiltonian system in Eq. (3) are

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} H_{\lambda x} & H_{\lambda\lambda} \\ -H_{xx} & -H_{x\lambda} \end{pmatrix}\begin{pmatrix} \delta x \\ \delta\lambda \end{pmatrix} \tag{12}$$

We assume that variational equation for the Hamiltonian system has an exponential dichotomy. The Hamiltonian nature of the above variational equation further dictates that $k = n$ and that σ and μ can be taken as equal.

A transformation that one-way decouples the contracting and expanding components of the motion can be achieved by a basis of the form

$$\begin{pmatrix} \delta x \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix}\delta x + \begin{pmatrix} 0 \\ I \end{pmatrix} w_u \tag{13}$$

for expressing the phase tangent vector $(\delta x, \delta\lambda)$ in the new coordinates $(\delta x, w_u)$, where P is the positive semi-definite solution to the differential Riccati equation

$$\dot{P} = -PH_{\lambda x} - H_{x\lambda}P - PH_{\lambda\lambda}P - H_{xx} \quad (14)$$

A boundary condition for P will be specified later.

For the general linear time-varying coordinate transformation $\delta p = Av$, the transformed variational equations in the new coordinates are

$$\dot{v} = (BJA - B\dot{A})v \quad (15)$$

where the columns of A are the basis vectors, $B = A^{-1}$, and $J = \frac{\partial f}{\partial p}$ is the Jacobian matrix. For the specific transformation introduced in Eq. (13), we have

$$A = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \quad (16)$$

$$B = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \quad (17)$$

In terms of the new coordinates, the variational equations are

$$\begin{pmatrix} \dot{\delta x} \\ \dot{w}_u \end{pmatrix} = \begin{pmatrix} H_{\lambda x} + H_{\lambda\lambda}P & H_{\lambda\lambda} \\ 0 & -(H_{\lambda x} + H_{\lambda\lambda}P)^T \end{pmatrix} \begin{pmatrix} \delta x \\ w_u \end{pmatrix} \quad (18)$$

Note that a tangent vector $(\delta x, w_u)$ for which w_u is zero initially will remain in the subspace given by span $\begin{pmatrix} I \\ P \end{pmatrix}$, when propagated along an extremal trajectory according to the variational equations, due to the zero off-diagonal block.

4. SOLUTION OF TWO TIME-SCALE HBVPs

We consider the situation where the length of the time interval, boundary conditions and dynamics are such that the optimal trajectory in the phase space – the (x, λ) space – has a boundary-layer type two time-scale structure. The boundary-layer type, two time-scale behavior implies that there is a splitting of the Hamiltonian vector field into slow, fast-stable and fast-unstable components.

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = h_{slow} + h_{fs} + h_{fu} \quad (19)$$

(See Fenichel (1979) for the geometry of two time-scale systems. His theory is also discussed in Mease (1995).) We do not consider the case of fast oscillatory behavior that persists throughout the time interval or the presence of turning points. The trajectory has three qualitatively distinct segments. The trajectory exhibits fast changes near the initial and final times, referred to as the initial and terminal boundary layers, respectively. The fast-stable component of the vector field will decay to zero quickly in forward time and only influence the trajectory in the initial boundary layer. The fast-unstable component of the vector field will decay to zero quickly in backward time and only influence the trajectory in the terminal boundary layer. Over most of the time interval, i.e., in between the boundary layers, the trajectory will be dictated by the slow component of the vector field. The terms slow and fast describe not only the relative rates of change of the trajectory in various regions of the time interval, but also the rates relative to the length of the time interval. The splitting that is appropriate for a given time interval may not be so for a longer or shorter time interval.

If the Hamiltonian system can be represented in *standard form*, the solution method of matched asymptotic expansions can be applied (see Kelley (1973) and Ardema (1976) and references therein). Our objective is to develop a solution method that applies to a two time-scale Hamiltonian system of arbitrary form. One strategy would be to seek a phase variable transformation that puts the system in standard form. Our strategy is to seek a (state-costate dependent) basis for the vector field that yields the desired splitting. I.e., we continue to use (x, λ) for the phase variables, but define alternative phase rate variables to $(\dot{x}, \dot{\lambda})$ that at least approximately yield the desired splitting. The dichotomy transformation plays a central role.

4.1. $h_{slow} = 0$ *Case*

The simplest case to treat is when there is only fast behavior, i.e., the case $h_{slow} = 0$. The slow solution in this case is a constant equilibrium solution (x_e, λ_e) satisfying $h_{fs}(x_e, \lambda_e) = 0$ and $h_{fu}(x_e, \lambda_e) = 0$. Let us assume that $f(x_e, 0) = 0$. As pointed out for example by Kokotovic *et al.* (1986), the optimal control problem can be solved as two infinite horizon optimal regulator problems. For initial and terminal conditions inconsistent with the equilibrium solution, one optimal regulator drives the initial state to the equilibrium in forward time; another optimal regulator drives the terminal state to the equilibrium in backward time.

The Hamiltonian nature of the vector field and the boundary layer solution structure require the equilibrium to be of saddle type. A saddle type equilibrium has a stable manifold and an unstable manifold. (See Guckenheimer and Holmes (1983) for background.) The approximate solution we are proposing is constructed by piecing together (not smoothly) a trajectory on the stable manifold and a trajectory on the unstable manifold. These trajectories are the solutions for particular initial and terminal conditions to the initial and terminal regulator problems respectively. These solutions can be computed as follows.

4.2. *Backward Sweep Procedure for Infinite-Horizon Nonlinear Regulator Problem*

To compute the solution to the initial regulator problem, the strategy is to establish an extremal on the stable manifold using knowledge of the stable eigenspace at the equilibrium and then to successively compute neighboring extremals on the stable manifold, leading to the one that satisfies the prescribed initial condition. Each required intermediate initial condition is computed as a perturbation in the tangent space to the stable manifold at the previous initial condition. This tangent space is identified using the solution to the differential Riccati equation with boundary condition consistent with the stable eigenspace at the equilibrium. Error growth during the forward integration of extremals on the stable manifold is avoided by removing the unstable component of the vector field. A description of the algorithm follows. The trajectory on the unstable manifold is computed using the same method and reversing time so that the terminal condition is considered an initial condition.

Step 1. *Getting on the stable manifold.* Perturb slightly the phase vector (state-costate vector) away from the equilibrium in the stable eigenspace.

Step 2. *Initial backward integration.* Integrate from this point backward in time. Stop the integration when the phase is as close to the hyperplane $x = x_0$ as possible. Also integrate simultaneously the differential Riccati equation backward from the condition $P(t_f^b) = \bar{P}$ where $\bar{P}$ is the solution to the algebraic Riccati equation with matrices evaluated at the equilibrium and t_f^b is an arbitrary starting time for the integration.

Step 3. *Forward Integration.* Let the time range for the backward integration be t_f^b to t_0^b, with $t_f^b > t_0^b$. Adjust the "initial" state $x(t_0^b)$ by a small increment δx toward x_0 and adjust the "initial" costate by $\delta\lambda = P(t_0^b)\delta x$. Integrate forward

$$\frac{d}{d\tau}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & 0 \end{pmatrix}\begin{pmatrix} H_\lambda \\ -H_x \end{pmatrix} \tag{20}$$

until the phase is within a specified distance of the equilibrium. The Riccati matrix has been saved from the backward sweep and is parametrized by time along the backward trajectory. To the new initial condition for (x, λ) is attached the time $t_0^f = 0$. For the forward integration, use $P^f(t^f) = P^b(t_0^b + t^f)$ for $t^f \le t_f^b - t_0^b$. If the forward integration requires a longer (in time) integration than did the preceding backward sweep, then use $P(t^f) = P(t_f^b) = \bar{P}$ for $t^f > t_f^b$.

Step 4. *Backward integration.* Integrate the Riccati equation backward from the point $P(t_f) = P_\infty$ along the phase trajectory saved from the preceding forward integration. The time interval is the same as for the preceding forward integration.

Step 5. *Convergence check.* Check for convergence. If not converged, repeat Steps 3 and 4.

Remarks. As in the traditional backward sweep method (see e.g. Bryson and Ho, 1975), the phase vector is adjusted by stable forward integration, using directional information supplied by the Riccati matrix that is obtained by stable backward integration. Some special features have been added to exploit the phase space geometry of the nonlinear regulator. The backward sweep is initialized on the stable manifold using eigenvalue/eigenvector information at the known equilibrium point. Except for the initial backward integration, the phase differential equations are only integrated in the forward direction with the unstable part of the vector field removed. Note that the projection matrix in the above equation eliminates the unstable component of the vector field, so that if the phase vector is perturbed off the stable manifold, the error will not be amplified by forward integration. Regarding the initial backward integration, the stable manifold is attracting in backward time, so that with sufficient care the computed trajectory should lie on the stable manifold. This is all that is required from the initial backward sweep. Also, the integration time interval is not held fixed during the iterative process. It is potentially important not to constrain the time interval to some arbitrary length. Note that if x is scalar then steps 1 and 2 generate the optimal solution; iteration is unnecessary.

A variation of this method and some numerical experience with it are presented in Rao and Mease (1995a,b).

4.3. $h_{slow} \neq 0$ *Case*

The geometry of the phase space in the neighborhood of a boundary layer type extremal trajectory is more complex in this case (Fenichel, 1979). The slow segment of the trajectory is not a constant equilibrium, but rather neighbors a segment of a trajectory on an invariant submanifold of the phase space, called the slow manifold. Under the assumptions we have made, the trajectory segment on the slow manifold is the transverse intersection of fast-stable and fast-unstable manifolds. The boundary layer type extremal trajectory begins slightly off the fast-stable manifold and follows it quickly towards the slow manifold. The trajectory then progresses slowly alongside the slow manifold. Close to the final time, the trajectory quickly follows the unstable manifold to the terminal condition slightly off the unstable manifold. For given boundary conditions on the state, as the length of the time interval increases, the initial and terminal boundary layer segments of the trajectory lie closer and closer to the fast-stable and fast-unstable manifolds, respectively.

Thus the approximate solution we seek has its initial boundary layer segment on a stable manifold and its terminal boundary layer segment on an unstable manifold, these manifolds corresponding to a particular trajectory on the slow manifold. There is not an exact solution with this behavior, because if a trajectory begins on the fast-stable manifold, it will quickly approach the slow manifold and then never leave. The same property holds in backward time for a trajectory that begins on the fast-unstable manifold. This is why the exact solution must begin slightly off the fast-stable manifold and terminate slightly off the fast-unstable manifold. The initial costate value primarily controls the initial boundary layer segment so that it follows the fast-stable manifold. However, to the initial costate value that would place the initial boundary-layer segment on the fast-stable manifold must be added a small increment that ensures that the trajectory will not remain on the slow manifold, but instead leave at the appropriate time following the unstable manifold to meet the specified terminal condition. It is the control of the terminal boundary-layer via this small increment in the initial costate that makes the two time-scale HBVP difficult to solve. Moreover, the required increment in the initial costate value would in turn require an unreasonable or perhaps unachievable level of precision in controlling a physical system. Our approach thus makes no attempt to compute the small increment.

Computing the extremal trajectory that satisfies the initial state condition, begins on the fast-stable manifold, continues on that manifold to the slow manifold, and then proceeds on the slow manifold is challenging due to the everpresent fast-unstable component of the vector field, h_{fu}. With an appropriate basis to split the vector field into the components h_{slow}, h_{fs} and h_{fu}, the proposed procedure is to choose λ_0 such that $h_{fu}(x_0, \lambda_0) = 0$ to begin and continue on a fast-stable manifold. Once the slow manifold is reached, as indicated by $h_{fs} = 0$, the conditions $h_{fs} = 0$ and $h_{fu} = 0$ are enforced to remove stiffness so that the integration step size can be increased. The procedure is reversed to compute an extremal trajectory from the terminal state condition that begins on an unstable manifold and is integrated backward in time. The condition $h_{fs} = 0$ does not fully determine the initial costate value; similarly the condition $h_{fu} = 0$ does not fully determine the terminal costate value. The remaining freedom in the initial and terminal costate values is used to achieve matching of the forward and backward trajectories, in the spirit of the method of matched asymptotic expansions. An important distinction is that this method does not require the Hamiltonian BVP to be given in standard form.

A basis for achieving the desired splitting approximately can be constructed from the eigenvectors of the local Jacobian matrix. On or near the slow manifold, this approximation is quite accurate (see discussion in Mease, 1995). Once a point on the slow manifold is located, the backward sweep method described above can be used to extend the basis away from the slow manifold. As mentioned above, on the slow manifold the fast components of the vector field should be zero whereas off the slow manifold the fast components are nonzero and (except in a small adjacent layer) much larger in magnitude than the slow component. Thus one can look for points in phase space where the magnitude of the vector field is relatively small, in the spirit of the bounded derivative method of Kreiss (1979). A variation of this approach is to use the local eigenvalues and eigenvectors to obtain approximations of h_{fs} and h_{fu} and look for points where they are both zero.

5. CLOSING REMARKS

The geometric structure of Hamiltonian systems and the notion of a dichotomy have guided the conceptualization of an indirect solution approach for two time-scale trajectory optimization problems. An algorithm has been given for a subclass of such problems.

6. ACKNOWLEDGMENT

This research was sponsored by the National Science Foundation. Valuable discussions with Prof. S.-H. Lam, Princeton University, during the course of this research are gratefully acknowledged.

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