Identifying Time-Scale Structure for Simplified Guidance Law Development

Sanjay Bharadwaj,[*] Mengjin Wu,[†] Kenneth .D. Mease[‡§]
Dept. of Mechanical and Aerospace Engineering
University of California, Irvine, CA 92697

Abstract

Guidance law development for aerospace vehicles can be significantly simplified by reducing the order of the vehicle dynamic model. Identifying time-scale separation in the system dynamics is important in this regard. This paper introduces a systematic means of identifying the time-scale structure in a nonlinear dynamical system by using regional Lyapunov exponents. Regional Lyapunov exponents and associated vectors are used to analyze the linear variational equations associated with a nonlinear system, in order to draw conclusions about the time-scale structure in the nonlinear system. Examples are used to illustrate the the time-scale information provided by the regional Lyapunov exponents. The time scale structure in an example of closed-loop entry dynamics is investigated using the regional Lyapunov exponents. The exponents and their associated directions identify energy as the slow variable and radial distance and flight path angle as fast variables and thus provide the information needed for reduced order guidance law development.

Introduction

Guidance problems for aerospace vehicles generically involve nonlinear dynamics, state and control constraints, and finite time intervals, and as such they do not lend themselves directly to much of the existing control design methodology. Effective guidance law development is greatly facilitated if the order of the vehicle dynamic model can be reduced. Using a rigid body model, neglecting actuator dynamics, and further assuming that the attitude motion is much faster than the translational motion so that attitude variables can be treated as controls for the translational motion, significantly reduces the order of the dynamic model and is often sufficiently accurate for guidance law design. Even so, developing an effective guidance law remains a formidable challenge due to the features mentioned above.

Further order reduction may be possible if there is time-scale separation in the translational dynamics. For example the energy state approximation[1] has been used effectively for some guidance problems. If the time-scale separation is understood to the point of knowing how to group the state variables according to the time-scales on which they evolve, then the singular perturbation method provides a formalism for constructing an accurate approximation from solutions to lower-order subproblems.[2,3] Calise[4] proposed to reduce the guidance law design task to a sequence of scalar design subtasks by hypothesizing that for the closed-loop translational dynamics, each state variable would evolve on a distinct time-scale. This approach has been referred to as forced singular perturbations since the time-scale separation is forced into the system using feedback control. While forced singular perturbations simplifies guidance law design, it may incur a control effort penalty.

What has been missing is a reliable and systematic means of identifying time-scales and the corresponding structure in nonlinear dynamic models, i.e., the appropriate counterpart to eigenvalues and eigenvectors for linear, time-invariant dynamic models. It seems that the appropriate focal point for identifying the time-scale structure of a nonlinear system is the behavior of the solutions to the variational equations along trajectories of the nonlinear system. The variational equations can be viewed as linear time-varying : the variational system matrix actually depends on the state of the nonlinear system, but along a trajectory of the nonlinear system this dependence can be treated as a dependence on time. Lyapunov introduced characteristic exponents and associated direction vectors for linear time-varying

[*] Graduate Student, Student Member AIAA.
[†] Graduate Student, Student Member AIAA.
[‡] Associate Professor, Associate Fellow AIAA.

systems. Lyapunov exponents give an average rate of expansion or contraction over an infinite time interval, when the appropriate limits exist. For guidance problems, we are interested in time-scales associated with a finite-time segment of a trajectory. Recently, Wiesel[6] has introduced regional Lyapunov exponents. These are defined similar to Lyapunov exponents, except that they give average rates of contraction and expansion over a finite-time interval.

In this paper, we develop an approach to identifying the time-scale structure of a nonlinear dynamic model based on the use of regional Lyapunov exponents. The paper is organized as follows. The general form of the nonlinear dynamic model is presented, along with the associated linear variational equations. The classical and regional Lyapunov exponents are defined. The regional Lyapunov exponents are computed for a simple model to illustrate the information they provide, the interpretation of this information, and the considerations for choosing the time-interval over which the rates of contraction and expansion are averaged. We then relate the information regarding the structure of the solutions to the variational equations to the time-scale structure of the nonlinear system in the context of an example of closed-loop entry dynamics.

Linear Variational System Associated With A Nonlinear Dynamical System

Consider a nonlinear system of the form

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables and $u \in \mathbb{R}^m$ is the vector of control variables. We are interested in investigating the time-scale structure of the above system with a control which is given by either a feedback law of the form $u = k(x)$ or an open-loop program of the form $u(t) = k(x(t), \lambda(t))$ as a result of solving an optimal control problem, where $\lambda \in \mathbb{R}^n$ is the vector of co-state variables satisfying $\dot{\lambda} = -\frac{\partial H}{\partial x}(x, \lambda)$, and $H(x, \lambda)$ is the optimal Hamiltonian. In the former case, the space of interest over which trajectories of the closed loop system $\dot{x} = f(x, k(x))$ evolve is the state space $\mathbb{R}^n$ whose points are represented by x. In the latter case, the space of interest is the phase space $\mathbb{R}^{2n}$ whose points are represented by (x, λ).

In either case, the time-scale structure can be analyzed by studying the system of variational equations which are obtained by linearizing the nonlinear system about a trajectory in the appropriate space of interest. The variational equations associated with the system, $\dot{x} = f(x, k(x))$, are given by

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x)\delta x \tag{2}$$

whereas the variational equations associated with the system, $\dot{x} = f(x, u(t))$ where $u(t)$ is obtained by solving an optimal control problem as described in the latter case, are given by

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{\lambda} \end{pmatrix} = \begin{pmatrix} H_{\lambda x} & H_{\lambda\lambda} \\ -H_{xx} & -H_{x\lambda} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta\lambda \end{pmatrix} \tag{3}$$

where terms like $H_{x\lambda}$ denote the second partial derivatives of the optimal Hamiltonian with respect to the variables indicated in the subscript. The trajectory (in the state space or the phase space) about which the linearization is carried out is naturally parameterized by time, and therefore the variational equations can be analyzed as a linear-time-varying (LTV) system.

An analysis of the time-scale structure of the trajectories of the variational equations yields information about the behavior of trajectories of the nonlinear system in the neighborhood of the reference trajectory to which the variational equations correspond. Therefore, the time-scale structure in the variational equations yields information about the time-scale structure of the nonlinear system. The following section describes measures for characterizing time scales in LTV systems.

Measures of Time-Scale Separation

In linear time invariant (LTI) systems of the form $\dot{x} = Ax$, the complete time-scale information can be obtained by examining the eigenvalues $\lambda_i, 1 \le i \le n$, and the corresponding eigenvectors v_i. The difference in magnitude of the real part of the eigenvalues $|Re(\lambda_i)|$ measures the separation in time-scales. For example, if $|Re(\lambda_i)| >> |Re(\lambda_j)|$, then states in the subspace spanned by v_i evolve much faster than those in the subspace spanned by v_j (provided $|Im(\lambda_j)|$ is not significantly larger than $|Im(\lambda_i)|$). Thus, the separation among $|Re(\lambda_i)|$-s determines the sets of eigenvectors whose spans decompose the state space into subspaces with distinct time-scale behavior.

We now consider LTV systems of the form

$$\dot{x} = A(t)x \ , x \in \mathbb{R}^n \tag{4}$$

where $A(t)$ is continuous and bounded. For LTV systems where $A(t)$ is periodic, the Floquet multipliers and the associated direction vectors can be used exactly analogous to eigenvalues and eigenvectors in the LTI case, to identify the time-scale structure. For general LTV systems, the asymptotic rates of contraction or expansion of solutions are characterized by the Lyapunov exponents.[5] The Lyapunov exponent associated with a solution $x(t, x_0)$ of the LTV system (4), where $x(t_0) = x_0$, is given by

$$\kappa(x_0) = \lim_{t\to\infty} \frac{\|x(t, x_0)\|}{t} \tag{5}$$

when the limit exists, and the direction vector associated with the Lyapunov exponent is x_0. Note that the Lyapunov exponents reduce to eigenvalues when applied to an LTI system. For simplicity consider a scalar LTI system $\dot{x} = \lambda_1 x$ with a solution $x(t, x_0) = \exp{(\lambda_1 t)} x_0$. The Lyapunov exponent $\kappa(x_0) = \lambda_1$.

In general, the LTV system (4) has at most n distinct Lyapunov exponents and at most n linearly independent associated direction vectors.[5] The Lyapunov exponents themselves are independent of the choice of the initial time t_0, but the associated directions depend on the choice of t_0. The separation in the Lyapunov exponents determines the time-scale separation in the LTV system. Analogous to the eigenvectors of the LTI system, the direction vectors associated with the Lyapunov exponents determine the decomposition of the state space into subspaces with distinct time-scale behavior. However since the direction vectors are dependent on the choice of t_0, they can be viewed as providing a time varying basis vectors which at each instant span the corresponding subspaces which separate the time-scale behavior.

The Lyapunov exponents are averages over an infinite time interval of the rates of contraction or expansion of solutions. For our purposes, it is averages over finite time intervals that are relevant. Wiesel[6] has defined regional Lyapunov exponents that measure the average contraction or expansion of solutions over a finite time interval. The regional Lyapunov exponents are defined as

$$\begin{aligned} \lambda_i &= \left(\tfrac{1}{t_f - t_0}\right) \ln\left(\tfrac{\|x_i(t_f)\|}{\|x_i(t_0)\|}\right) \\ &= \left(\tfrac{1}{t_f - t_0}\right) \ln\left(\tfrac{\|\Phi(t_f, t_0)x_i(t_0)\|}{\|x_i(t_0)\|}\right) \end{aligned} \tag{6}$$

where $\Phi(t, t_0)$ is the state-transition matrix of the LTV system (4). The associated directions at t_0 are given by $x_i(t_0)$ and hence at any time t they are given by $x_i(t) = \Phi(t, t_0)x_i(t_0)$. In order to determine the entire spectrum of Lyapunov exponents λ_i without an exhaustive trial and error procedure with arbitrary vectors at t_0, we need to, if possible, compute a linearly independent set of vectors $x_i(t_0)$ that correspond to the different time-scales. Such vectors can be obtained[6] by solving for the unit vectors $\|x(t_0)\| = 1$ that extremize

$$\left(\frac{1}{t_f - t_0}\right) \ln\left(\|\Phi(t_f, t_0)x(t_0)\|\right)$$

It can be shown that the eigenvectors of $\Phi(t_f, t_0)^T \Phi(t_f, t_0)$ are the solutions to this extremization problem and hence are the vectors $x_i(t_0)$, $i = 1, \ldots, n$ that we need to compute the spectrum of regional Lyapunov exponents.

We shall now, following Wiesel, show that with the knowledge of these regional Lyapunov exponents and their associated directions at t_0 (and hence for all time t), a coordinate transformation that diagonalizes the LTV system (4) (and hence decouples the different time-scales in the system) can be constructed.

Consider the directions $e_i = \frac{x_i}{\|x_i(t)\|}$

$$\begin{aligned} \dot{e}_i &= \left(\tfrac{1}{\|x_i\|^2}\right)\left(\dot{x_i}\|x_i\| - x_i \tfrac{d}{dt}(\|x_i\|)\right) \\ &= \tfrac{A(t)x_i}{\|x_i\|} - \tfrac{x_i}{\|x_i\|}\left(\tfrac{1}{\|x_i\|}\tfrac{d}{dt}(\|x_i\|)\right) \\ &= A(t)e_i - e_i\sigma_i(t) \end{aligned}$$

where

$$\sigma_i(t) = \frac{1}{\|x_i(t)\|}\frac{d}{dt}(\|x_i(t)\|) \tag{7}$$

is the normalized rate of change of the magnitude of $\|x_i(t)\|$. Therefore, the evolution of all the directions $e_i(t), 1 \le i \le n$ together is described by the matrix differential equation

$$\dot{E}(t) = A(t)E(t) - E(t)\Sigma(t) \tag{8}$$

where $\Sigma(t) = diag(\sigma_i(t))$ and

$$E(t) = \begin{bmatrix} | & | & | \\ e_1(t) & \cdots & e_n(t) \\ | & | & | \end{bmatrix} \tag{9}$$

We now consider the time-varying coordinate transformation $x(t) = E(t)\xi(t)$ so that the dynamics in the new coordinates $\xi(t)$ are given by

$$\dot{\xi} = \Sigma(t)\xi \tag{10}$$

which are in diagonal form where the different time-scales are decoupled. Now, note from Eq. (7), it can be shown that

$$\frac{\|x_i(t_f)\|}{\|x_i(t_0)\|} = \int_{t_0}^{t_f} \sigma_i(t)dt$$

so that the regional Lyapunov exponent in Eq. (6) is

$$\begin{aligned} \lambda_i &= \left(\tfrac{1}{t_f - t_0}\right) \textstyle\int_{t_0}^{t_f} \sigma_i(t)dt \\ &= \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \frac{1}{\|x\|} \frac{d(\|x\|)}{dt}\, dt \end{aligned} \tag{11}$$

This shows that the regional Lyapunov exponent is indeed the average over the time interval $[t_0, t_f]$ of the rate of change of the magnitude of $x_i(t)$.

Illustrative Example

Consider an LTV system $\dot{x} = A(t)x$, where $A(t) = [a_{ij}(t)]$

$$\begin{aligned} a_{11}(t) &= -5 + 0.2\sin(t) \\ a_{12}(t) &= \frac{\sin(3t^2)}{t + 0.01} \\ a_{13}(t) &= 2.5\sin(t)\exp(-0.01t) \\ a_{21}(t) &= 1 + \exp(-0.2t) \\ a_{22}(t) &= -15\exp(-0.002t) \\ a_{23}(t) &= \frac{t}{2 + t + \cos(t^3)} + 3 \\ a_{31}(t) &= \frac{t}{t^2 + 1} \\ a_{32}(t) &= \exp(-0.005t) + 1 \\ a_{33}(t) &= \frac{-t}{t + 1} \end{aligned}$$

With $t_f = 1$ the regional Lyapunov exponents are $\lambda_1 = -4.9734$, $\lambda_2 = -15.5391$, $\lambda_3 = +0.3126$. The associated direction vectors are $e_1(t_0) = [-0.9964\ ,\ -0.0107\ ,\ 0.0836]^T$, $e_2(t_0) = [-0.0214\ ,\ 0.9915\ ,\ -0.1285]^T$ and $e_3(t_0) = [0.0815\ ,\ 0.1298\ ,\ 0.9882]^T$. This implies that there are 3 distinct time-scales over the time interval $[0, 1]$. $e_2(t)$ is the fastest direction and is stable as indicated by the negative sign of λ_2. $e_1(t)$ is the slowest direction and is unstable as indicated by the sign of λ_3.

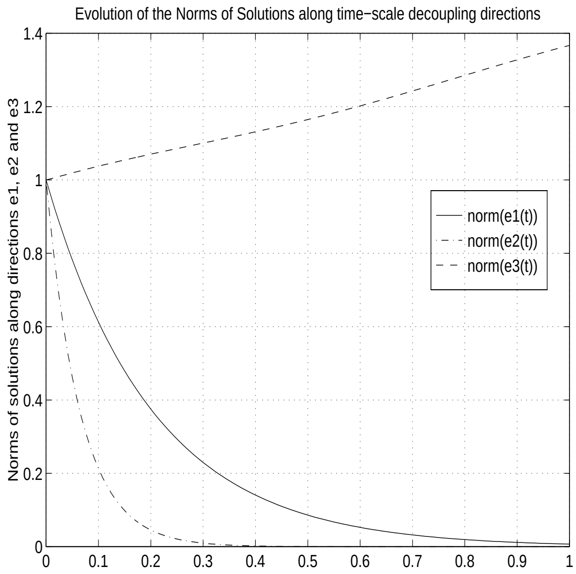

Fig. 1: $\|e_1(t)\|$,$\|e_2(t)\|$ and $\|e_3(t)\|$ vs. Time for $t_f = 1$ sec.

Fig. 1 shows the evolution of norms of solutions to this LTV system along the directions $e_1(t), e_2(t)$ and $e_3(t)$. Fig. 2 shows the normalized instantaneous rates of change of the solutions along the above directions, i.e., $\sigma_1(t), \sigma_2(t)$ and $\sigma_3(t)$. Both these show clearly the time-scale separation present in the system over the interval $[0, 1]$.

With $t_f = 15$, the regional Lyapunov exponents are $\lambda_1 = -2.9061$, $\lambda_2 = -2.8139$, $\lambda_3 = -0.2736$ and their associated directions which decouple the time-scales are $e_1(t_0) = [0.6248\ ,\ -0.7791\ ,\ 0.0516]^T$, $e_2(t_0) = [-0.7716\ ,\ -0.6133\ ,\ 0.1437]^T$ and $e_3(t_0) = [0.0803\ ,\ 0.1298\ ,\ 0.9883]^T$. As in the previous two plots, Figs. 3 and 4 show the evolution of the norms of solutions along these new directions and their normalized rates of change. We can see that over the interval $[0, 15]$, there is no significant time-scale separation between directions e_1 and e_2. However, vectors in $span\{e_1, e_2\}$ decay much faster than vectors along e_3. Also, note that over the interval $[0, 15]$, e_3 is now a stable direction as indicated by the negative sign of λ_3.

Application to Closed-Loop Entry Dynamics

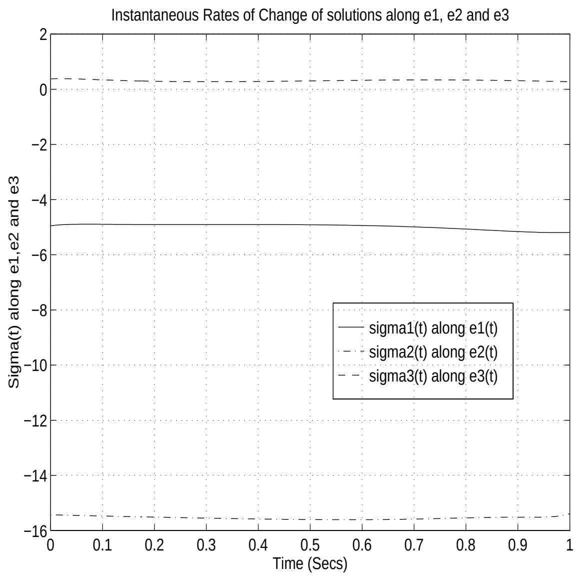

Fig. 2: $\sigma_1(t)$,$\sigma_2(t)$ **and** $\sigma_3(t)$ **vs. time for** $t_f = 1$ **sec.**

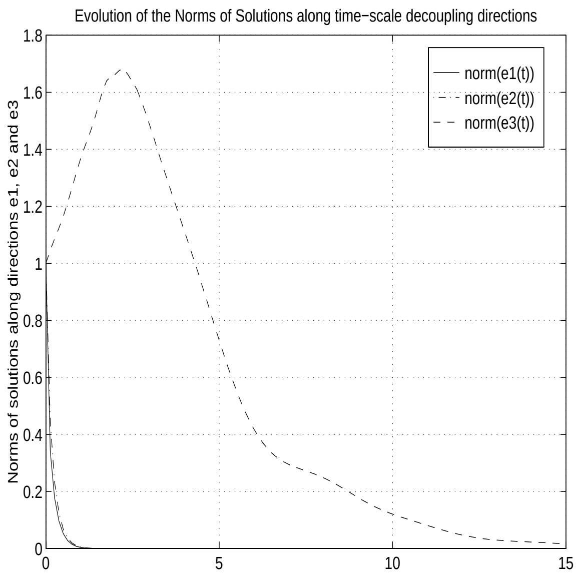

Fig. 3: $\|e_1(t)\|$,$\|e_2(t)\|$ **and** $\|e_3(t)\|$ **vs. Time for** $t_f = 15$ **sec.**

In this section we shall use the regional Lyapunov exponents to uncover the time-scale structure in an example of closed loop entry dynamics. The vertical-plane dynamics of a gliding aerospace vehicle are given by the equations

$$\begin{array}{rcl} \dot{r} &=& V\sin\gamma \\ \dot{V} &=& -D - g\sin\gamma \\ \dot{\gamma} &=& (1/V)\left(L - (g - V^2/r)(\cos\gamma)\right) \end{array} \tag{12}$$

where r is the radial distance from the center of the earth, V is the earth-relative velocity and γ is the flight path angle. The specific drag force $D = (0.5\rho(r)V^2SC_D)/m$ where C_D, the coefficient of drag, is taken to be constant and m is the mass of the vehicle during entry. $\rho(r) = \rho_0\exp(-(r-r_0)/H)$ is an exponential model for the density as a function of altitude.

Lift is taken to be the control which is used to track a reference altitude vs. velocity profile. The feedback control law

$$\begin{aligned} L &= (g - V^2/r)\cos\gamma + \left(\frac{1}{\cos\gamma}\right)(D + g\sin\gamma)\sin\gamma \\ &\quad + \left(\frac{1}{\cos\gamma}\right)\left[\ddot{r}_{ref} - k_p(r - r_{ref}) - k_d(\dot{r} - \dot{r}_{ref})\right] \end{aligned}$$

is used to track a reference radial distance versus velocity profile. Note that the error terms $(r - r_{ref})$ and $(\dot{r} - \dot{r}_{ref})$ are referenced to the current velocity.

The variational equations of the closed-loop system are then given by

$$\begin{bmatrix} \delta\dot{r} \\ \delta\dot{V} \\ \delta\dot{\gamma} \end{bmatrix} = A(r_{ref}(t), V_{ref}(t), \gamma_{ref}(t)) \begin{bmatrix} \delta r \\ \delta V \\ \delta\gamma \end{bmatrix}$$

where $A = [a_{ij}]$,

$$\begin{array}{rcl} a_{11}(t) &=& 0 \\ a_{12}(t) &=& \sin\gamma_{ref} \\ a_{13}(t) &=& V_{ref}\cos\gamma_{ref} \\ a_{21}(t) &=& D_{ref}/H \\ a_{22}(t) &=& -2D_{ref}/V_{ref} \\ a_{23}(t) &=& -g\cos\gamma_{ref} \\ a_{31}(t) &=& \left(\dfrac{-\left(D_{ref}\sin\gamma_{ref}/H + k_p\right)}{V_{ref}\cos\gamma_{ref}}\right) \\ a_{32}(t) &=& 0 \\ a_{33}(t) &=& \dfrac{D_{ref} + 2g\sin\gamma_{ref}}{V_{ref}} - k_d \end{array}$$

These are used to investigate the time-scale structure of the nonlinear system.

The regional Lyapunov exponents and their associated direction vectors for the above LTV system over the time interval $[0, 750]$ seconds are $\lambda_1 = -5354.2$, $\lambda_2 = -5665.5$, $\lambda_3 = -2.2659$

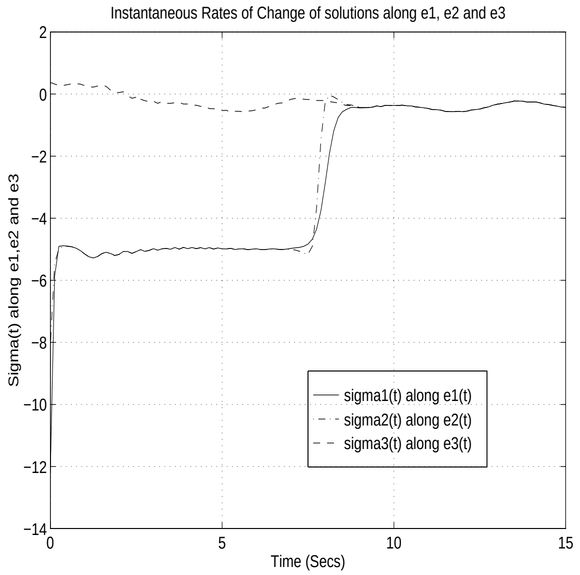

Fig. 4: $\sigma_1(t)$,$\sigma_2(t)$ and $\sigma_3(t)$ vs. time for $t_f = 15$ sec.

and $e_1(t_0) = [-0.9288\ ,\ 0.0613\ ,\ -0.3656]^T$, $e_2(t_0) = [0.3661\ ,\ -0.0029\ ,\ -0.9306]^T$, $e_3(t_0) = [0.0581\ ,\ 0.9981\ ,\ 0.0198]^T$. From these, we can now infer that in the variational LTV system, perturbations in all directions decay. Therefore, for the nonlinear dynamics (12), trajectories neighboring the reference converge to it. The regional Lyapunov exponents also indicate that vectors in $span\{e_1, e_2\}$ decay much faster the vectors in $span\{e_3\}$. Figs. 5 and 6 show the time-scale separation graphically.

In Fig. 7, the feedback tracking of the reference altitude vs. velocity profile is shown. Also shown at each point along the reference trajectory are the projections of the basis vectors $e_1(t)$ and $e_3(t)$ onto the $r - V$ plane. The fast direction is indicated by the double arrowhead. Curves of constant energy are also plotted. It can be seen that the fast direction is tangent to the constant energy curves. Simulations of the closed loop system show that perturbations in altitude away from the reference quickly decay along almost constant energy lines. Thus the regional Lyapunov exponents and associated direction vectors have successfully identified energy as a slow variable, while altitude and flight path angle are fast variables. It is important to point out that, although the example was concocted such that energy is a slow variable, the method is not at all biased or restricted to this outcome.

Conclusions and Comments

A systematic means of identifying time-scales for nonlinear dynamical systems using the regional Lyapunov exponents was introduced. Numerical examples were used to illustrate the time-scale information that these regional Lyapunov exponents provide. This time-scale analysis was applied to the closed-loop entry guidance dynamics and shown to identify the energy as a slow variable and radial distance and flight path angle as fast variables. This is the information needed for reduced order guidance law development.

References

[1] Bryson A.E., Desai, M.N.,and Hoffman, W.C., “The Energy-state Approximation in Performance Optimization of Supersonic Aircraft”,*Journal of Aircraft*, Vol 6, pp. 841–846, 1969.

[2] Kelley, H.J., “Aircraft Maneuver Optimization by Reduced Order Approximation”,in Leondes, C.T. (ed.), *Control and Dynamic Systems*, Vol. 10, Academic Press, 1973.

[3] Ardema, M.D, and Rajan, N., “Slow and Fast Variables for Three-Dimensional Flight Dynamics”, *Journal of Guidance, Control and Dynamics*, Vol. 8, No. 4, pp. 532–535, 1985.

[4] Calise, A.J., “Singular Perturbation Methods for Variational Problems”, *IEEE transactions on Automatic Control*, Vol. 21, pp. 345–353, 1976.

[5] Coppel, W.A., “Dichotomies in Stability Theory”, *Springer-Verlag*, 1978.

[6] Wiesel, W.E., “Modal Feedback Control on Chaotic Trajectories”, *Physical Review E*, Vol. 49, No. 3, Mar., 1994.

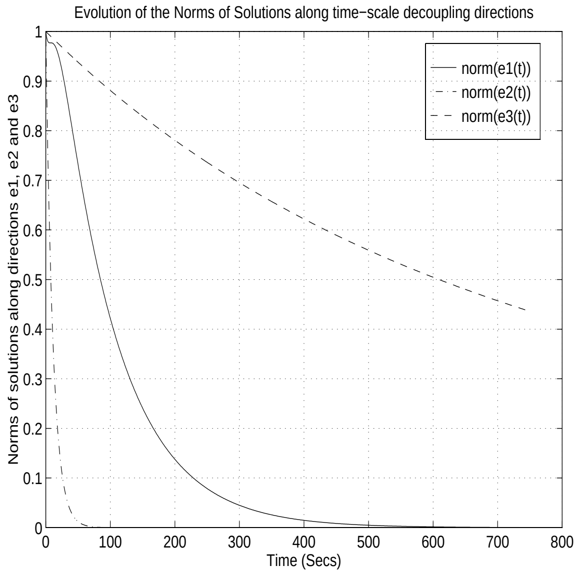

Fig. 5: $\|e_1(t)\|$,$\|e_2(t)\|$ and $\|e_3(t)\|$ vs. Time for $t_f = 750$ sec.

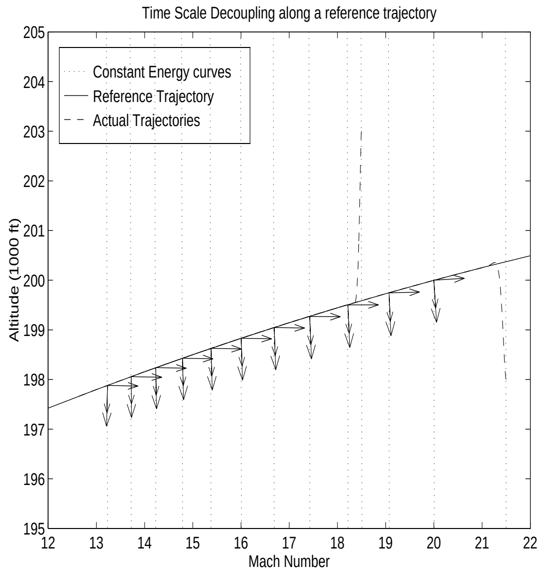

Fig. 7: Tracking of the reference trajectory and Time-Scale decoupling along the reference trajectory

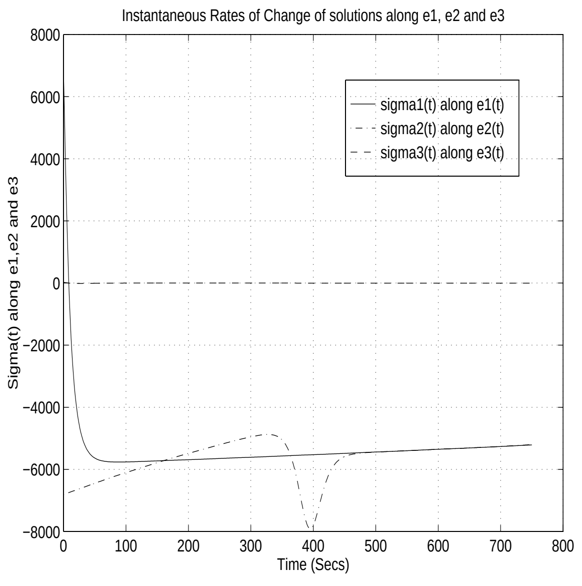

Fig. 6: $\sigma_1(t)$,$\sigma_2(t)$ and $\sigma_3(t)$ vs. time for $t_f = 750$ sec.