AIAA-95-3262-CP

A New Method for Solving Optimal Control Problems

Anil Vithala Rao* and Kenneth D. Mease†
Department of Mechanical and Aerospace Engineering
University of California - Irvine
Irvine, CA 92717

Abstract

A new method for solving optimal control problems is introduced. This method is a derivative of the computational singular perturbation methodology for initial value problems. The objective is to suppress the dynamics that do not contribute significantly to the optimal motion, but cause difficulty in computing the optimal solution. This is done by expressing the state-adjoint rate vector in terms of special basis vectors. These basis vectors are chosen so that they identify the directions of the insignificant but troublesome dynamics. As an initial step in the development of the method, we consider the class of nonlinear infinite horizon problems for which the optimal trajectory lies in the stable manifold of a saddle-point equilibrium in the state-adjoint space. It is shown that if an appropriate basis, called a modal basis, were known, the effect of the unstable dynamics could be eliminated without error, i.e., the initial condition on the stable manifold could be determined. The trajectory computed by integrating the state and adjoint forward in time then would be the *exact solution*. Since a modal basis can rarely be determined, an iterative procedure called the modified sweep method is developed that does not require a modal basis. If the iterative procedure converges, the solution to which it converges is an extremal solution of the optimal control problem.

Introduction

The solution of Hamiltonian Two-Point Boundary Value Problems (HTPBVP) arising in the context of optimal control is challenging due to the structure of their dynamics. Because of unstable dynamics, errors made guessing the unknown initial conditions tend to be amplified rather than attenuated.

HTPBVP's that are especially difficult to solve are those in which the dynamics evolve on two or more widely separated time-scales. A host of work has been done in finding solutions to trajectory optimization problems with multiple time-scale structure (see Ref. [1]-[7] for example). Khalil [8], et. al., address the general issue of developing solutions to multiple time-scale optimal control problems.

For two time-scale optimal control problems it is known that composite approximations for extremal solutions can be found by matching the initial and final fast (boundary layer) solutions and the slow outer solution. To zeroth-order, the initial and final boundary layer problems can be viewed as nonlinear infinite horizon regulator problems [8]. As a result, solving nonlinear infinite horizon regulator problems is a step in being able to handle two (or more) time-scale optimal control problems.

This paper introduces a new indirect method for numerically solving nonlinear infinite horizon optimal regulator problems for which the optimal trajectory lies on the stable manifold of a saddle-point equilibrium in the state-adjoint space. The method is a derivative of the computational singular perturbation (CSP) methodology for initial value problems [9, 10]. The method is related to the backward sweep method of [13], but with important differences. As a result, the method described here is called the *modified sweep method.*

HTPBVP Formulation

Lagrangian Optimal Control

We consider the problem of finding the control function u that minimizes the cost function

$$J = \int_0^{t_f} \mathcal{L}[x(t), u(t)]dt \tag{1}$$

*Graduate Student, Dept. of Mechanical and Aerospace Engineering, Princeton University, *In Absentia*.
†Associate Professor. Associate Fellow, AIAA. Author to whom correspondence should be sent.

subject to the dynamic constraint and initial condition

$$\dot{x} = f(x,u), \; x(0) = x_0 \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the *state*, $u(t) \in \mathbb{R}^m$ is the *control*, and $\mathcal{L}[x,u]$ is the *Lagrangian*. The final time is fixed. The *Hamiltonian* is given by

$$H = \mathcal{L} + \lambda^T f \tag{3}$$

where $\lambda(t) \in \mathbb{R}^n$ is the *adjoint* or *co-state* variable. From the Pontryagin minimum principle, the optimal control is found by

$$u^*(t) = \arg\min_u H \tag{4}$$

It is assumed that from Eq. (4) the optimal control u^* can be determined as a function of x and λ, i.e., $u^* = u^*(x,\lambda)$. Applying 1^{st}-order necessary conditions for optimality results in the following system of differential equations and boundary conditions:

$$\begin{aligned} \dot{x} &= \left[\frac{\partial H}{\partial \lambda}\right]^T, \; x(0) = x_0 \\ \dot{\lambda} &= -\left[\frac{\partial H}{\partial x}\right]^T, \; \lambda(t_f) = 0 \end{aligned} \tag{5}$$

where H now refers to the Hamiltonian evaluated on the optimal control $u^*(x,\lambda)$. Eq. (5) is a $2n$-dimensional HTPBVP. It can be shown that H is constant along a solution to Eq. (5). The space of points $\begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^{2n}$ in which trajectories to Eq. (5) evolve will be called the *Hamiltonian phase space* or simply the *phase space*; each vector $\begin{bmatrix} x \\ \lambda \end{bmatrix}$ is referred to as a *phase vector* and its time derivative $\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix}$ is called the *phase rate vector*.

Infinite Horizon Problem

The infinite horizon problem is characterized by taking the limit $t_f \to \infty$. We assume that the appropriate conditions [11] are satisfied such that a solution to the boundary value problem exists and approaches an equilibrium point asymptotically. We have that $\lim_{t\to\infty} \lambda(t) = 0$ and Eq. (5) becomes

$$\begin{aligned} \dot{x} &= \left[\frac{\partial H}{\partial \lambda}\right]^T, \; x(0) = x_0 \\ \dot{\lambda} &= -\left[\frac{\partial H}{\partial x}\right]^T, \; \lim_{t\to\infty} \lambda(t) = 0 \end{aligned} \tag{6}$$

Hamiltonian System Properties

Liouville's theorem [12] states that Eq. (6) satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \lambda}\right)^T - \frac{\partial}{\partial \lambda}\left(\frac{\partial H}{\partial x}\right)^T = 0 \tag{7}$$

In words, the divergence of the Hamiltonian vector field is zero. Liouville's theorem implies that all equilibria of an Hamiltonian vector field are either saddles or centers; there cannot be any sources or sinks. As a consequence of previous assumptions, all equilibria of Eq. (6) are saddle points. Each saddle point has an n-dimensional stable manifold and an n-dimensional unstable manifold. The stable manifold is the set of points S such that any trajectory emanating from a point on S will remain on S for all time and will approach the corresponding equilibrium asymptotically as $t \to \infty$. Figure 1 illustrates the structure of the flow near a saddle point at the origin for the case $n = 1$.

General Basis for Phase Rate

Rewriting the Derivative

It will be convenient to express the HTPBVP of Eq. (6) in the form

$$\dot{p} = G(p), \; \begin{bmatrix} p_1(t_0) = x_0 \\ \lim_{t\to\infty} p_2(t) = 0 \end{bmatrix} \tag{8}$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{9}$$

is used to denote the phase vector and

$$G(p) = \begin{bmatrix} \left[\frac{\partial H}{\partial \lambda}\right]^T \\ -\left[\frac{\partial H}{\partial x}\right]^T \end{bmatrix} \tag{10}$$

is the phase rate vector.

Since $G(p)$ is a vector, at any point p in the phase space the vector $G(p)$ can be expressed in terms of a known set of linearly independent *column* vectors $v_i(p) \in \mathbb{R}^{2n}$, $i = 1, 2, \ldots, 2n$, as

$$\begin{aligned} G(p) &= \sum_{k=1}^{2n} q_k v_k(p) \\ &= V(p) q \end{aligned} \tag{11}$$

where $V(p) = [v_1(p) \cdots v_{2n}(p)] \in \mathbb{R}^{2n \times 2n}$, $q = [q_1 \cdots q_{2n}]^T \in \mathbb{R}^{2n}$ and "^{T}" denotes matrix transpose. The vector q is called a *rate coordinate vector* and its components q_k, $k = 1, 2, ..., 2n$ are called *rate coordinates* of $G(p)$ in the basis given by the columns of the matrix $V(p)$. The inverse of $V(p)$ is given by $W(p) = V^{-1}(p) = \left[w_1^T(p) \cdots w_{2n}^T(p)\right]^T \in \mathbb{R}^{2n \times 2n}$ where $w_k(p) \in \mathbb{R}^{2n}$, $k = 1, 2, \ldots, 2n$ are *row* vectors. The vector q satisfies the relationship

$$q = W(p) \cdot G(p) \tag{12}$$

Now consider a basis field $V(p)$ that varies smoothly with p. Differentiate Eq. (12) along a trajectory of the differential equation of Eq. (8); the trajectory need not satisfy the boundary conditions. We obtain

$$\begin{aligned} \dot{q} &= \left[\dot{W}G + W\frac{\partial G}{\partial p}\dot{p}\right] \\ &= \dot{W}Vq + WJVq \\ &= \left[\dot{W}V + WJV\right]q \\ &= Zq \end{aligned} \tag{13}$$

where $J = \dfrac{\partial G}{\partial p} = \left[\dfrac{\partial G}{\partial x}\ \dfrac{\partial G}{\partial \lambda}\right] \in \mathbb{R}^{2n \times 2n}$ is the Jacobian matrix of $G(p)$ and $Z = \dot{W}V + WJV$, $Z \in \mathbb{R}^{2n \times 2n}$. Eq. (13) governs the evolution of the components of G in the chosen basis along the trajectory under consideration.

Modal Basis Vectors

A basis field V is said to be *modal with respect to stable and unstable behavior* if for each trajectory $p(\cdot)$ of the differential equation of Eq. (8) the matrix Z of Eq. (13) is block diagonal as

$$Z = \Lambda = \begin{bmatrix} \Lambda_u(t) & 0 \\ 0 & \Lambda_s(t) \end{bmatrix}. \tag{14}$$

where $\Lambda_u(t)$ and $\Lambda_s(t)$ are each of size $n \times n$, the state transition matrix $\Phi_u(t, 0)$ that arises from $\Lambda_u(t)$ is asymptotically stable as $t \to -\infty$, i.e., $\lim_{t\to-\infty} ||\Phi_u(t, 0)|| = 0$, and the state transition matrix $\Phi_s(t, 0)$ that arises from $\Lambda_s(t)$ is asymptotically stable as $t \to \infty$, i.e., $\lim_{t\to\infty} ||\Phi_s(t, 0)|| = 0$. When V is modal the following notations are used:

$$\begin{aligned} V &\to A \\ W = V^{-1} &\to B \\ q &\to h \end{aligned}$$

In the modal case the matrix A has the form

$$A = [A_u\ A_s] \tag{15}$$

where $A_u = [a_1 \cdots a_n]$ and $A_s = [a_{n+1} \cdots a_{2n}]$ and from Eq. (11) the vector $\dot{p} = G$ can be written as

$$\dot{p} = G = A_u h_u + A_s h_s \tag{16}$$

Because of the block-diagonal structure of Λ, the terms $A_u h_u$ and $A_s h_s$ of Eq. (16) are called *modes.* The subscript "u" is used to denote an unstable mode and the subscript "s" is used to denote a stable mode. Correspondingly, the vector $h(t)$ has the form $h(t) = \begin{bmatrix} h_u(t) \\ h_s(t) \end{bmatrix}$. The vectors $h_u(t)$ and $h_s(t)$ are called the *unstable and stable modal rate coordinates*, respectively. Remark: The stable and unstable modes can be further subdivided based on the asymptotic rates of contraction and expansion (analogous to modal decomposition for a linear time-invariant system using eigenvalues). However, in this paper we only consider stable and unstable modes and use the term modal to refer to this decomposition.

Every point $p \in \mathbb{R}^{2n}$ along the *minimizing solution* of Eq. (8) lies in the stable manifold. At these point the column-span of A_s corresponds to the tangent space to the stable manifold at p and $G(p)$ must lie in this tangent space. This is equivalent to saying that the unstable modal rate coordinates must be zero which, from Eq. (12) gives

$$h_u = B_u(p) \cdot G(p) \equiv 0 \tag{17}$$

where $B = \begin{bmatrix} B_u \\ B_s \end{bmatrix} = A^{-1}$. Eq. (17) can be used at $t = 0$ to to generate a boundary condition for the initial adjoint $\lambda(0) = \lambda_0$ by solving

$$B_u(p_0) \cdot G(p_0) = 0 \tag{18}$$

for $\lambda(0) = \lambda_0$. Then, $h_s(0)$ can be computed from

$$h_s(0) = h_{s0} = B_s(p_0) \cdot G(p_0) \tag{19}$$

If Eq. (17) is used as an *input* to Eq. (16) and the differential equation for $h_s(t)$, given by

$$\dot{h}_s = \left[\dot{B}_s A + B_s J A\right] \begin{bmatrix} h_u \\ h_s \end{bmatrix} \tag{20}$$

is integrated along with Eq. (16), the following system of $3n$ first-order differential equations result:

$$\begin{aligned} \dot{p} &= A_s h_s,\ p(0) = p_0 \\ \dot{h}_s &= \left[\dot{B}_s A + B_s J A\right] \begin{bmatrix} 0 \\ h_s \end{bmatrix},\ h_s(0) = h_{s0} \end{aligned} \tag{21}$$

Since the basis vectors are modal, the unstable component is removed from Eq. (21), the initial condition $p_0 = (x_0, \lambda_0)$ lies on the stable manifold, and the trajectory produced is the *exact solution* to Eq. (6).

Non-modal Basis Vectors

Non-modal basis vectors are of practical importance because it is generally infeasible to find a modal basis in which to represent the vector G. Let $p(\cdot)$ be a trajectory that satisfies the ODE of Eq. (8) but *not necessarily* the boundary conditions. At any point along $p(\cdot)$, the vector G can be expressed in terms of modal and non-modal quantities as

$$G = Ah = Vq \tag{22}$$

where V and q are the non-modal basis matrix and phase rate vector, respectively. From Eq. (22) it is easily verified that $q = WAh$. Letting $T = WA$ where

$$V = [V_u \ V_s],$$
$$W = \begin{bmatrix} W_u \\ W_s \end{bmatrix},$$
$$q = \begin{bmatrix} q_u \\ q_s \end{bmatrix},$$

and partitioning T so that $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, q_u and q_s can be written as

$$\begin{aligned} q_u(t) &= T_{11}h_u(t) + T_{12}h_s(t) \\ q_s(t) &= T_{21}h_u(t) + T_{22}h_s(t) \end{aligned} \tag{23}$$

The vectors $q_u(t)$ and $q_s(t)$ are referred to as *approximate stable and unstable rate vectors*, respectively. Now, the solutions for the modal rate vectors $h_u(t)$ and $h_s(t)$ are given as

$$\begin{aligned} h_u(t) &= \Phi_u(t,0)h_u(0) \\ h_s(t) &= \Phi_s(t,0)h_s(0) \end{aligned} \tag{24}$$

Upon substitution into Eq. (23), the solutions for the non-modal rate vectors $q_u(t)$ and $q_s(t)$ are given, respectively, by

$$\begin{aligned} q_u(t) &= T_{11}\Phi_u(t,0)h_u(0) + T_{12}\Phi_s(t,0)h_s(0) \\ q_s(t) &= T_{21}\Phi_u(t,0)h_u(0) + T_{22}\Phi_s(t,0)h_s(0) \end{aligned} \tag{25}$$

Suppose now that $p(t) = p^*(t) = \begin{bmatrix} x^*(t) \\ \lambda^*(t) \end{bmatrix}$, i.e., , the optimal solution. Since $h_u(t) \equiv 0$ on $p^*(t)$, we have

$$\begin{aligned} q_u^*(t) &= T_{12}^*\Phi_s^*(t,0)h_s^*(0) \\ q_s^*(t) &= T_{22}^*\Phi_s^*(t,0)h_s^*(0) \end{aligned} \tag{26}$$

Furthermore, the solution to Eq. (8) must decay to an equilibrium point from which it follows that

$$\lim_{t\to\infty} G(p^*(t)) = 0 \to \lim_{t\to\infty} A(p^*(t))h^*(t) = 0 \tag{27}$$

Since the matrix A is nonsingular by definition for all t, we have that

$$\lim_{t\to\infty} h^*(t) = 0 \to \lim_{t\to\infty} h_u^*(t) = 0 \text{ and } \lim_{t\to\infty} h_s^*(t) = 0 \tag{28}$$

Since T is a finite matrix for all time, the rate coordinate vectors $q_u(t)$ and $q_s(t)$ must satisfy

$$\lim_{t\to\infty} q_u^*(t) = \lim_{t\to\infty} q_s^*(t) = 0 \tag{29}$$

Eq. (29) states that while the non-modal rate coordinates are not identically zero, they decay to zero on the optimal solution (In fact, they decay to zero along any trajectory on the stable manifold). Furthermore, while the basis is non-modal, the pair (W_u, q_u) gives the same solution as that of Eq. (21). In terms of non-modal quantities, the ODE of Eq. (21) is given as

$$\begin{aligned} \dot{p} &= V_u q_u + V_s q_s, \ p(0) = p_0 \\ \dot{q}_s &= \left[\dot{W}_s V + W_s J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix}, \ q_s(0) = q_{s0} \end{aligned} \tag{30}$$

where the initial conditions $\lambda(0) = \lambda_0$ and $q_s(0) = q_{s0}$ are computed exactly as when using modal basis vectors, except non-modal quantities are used. It can be seen that the trajectory $p(\cdot)$ produced by integrating Eq. (30) is also the exact solution. A basis W and a rate coordinate q are said to be *consistent* if the solution of Eq. (30) is the exact solution of Eq. (8). While for a consistent pair (W_u, q_u) the solution of Eq. (30) is exact, Eq. (30) does not give any information about *finding* a consistent pair (W_u, q_u). In the next section an iterative approach will be described to obtain the exact solution to Eq. (8).

Modified Sweep Method

An iterative method arises directly as a consequence of the previous discussion. Its elements are described in the first four subsections and the method is stated concisely in the fifth subsection. In the final subsection, the method is compared with the well-known backward sweep method as described in Bryson and Ho [13].

Choosing Basis Vectors

The objective is to find a basis that is a reasonable approximation to a modal basis. One method for choosing basis vectors is motivated by how they would be determined for a linear time-invariant (LTI) system. Suppose we are given a system of the form

$$\dot{z} = Fz \tag{31}$$

where F is a constant matrix. It is well-known that the eigenvectors of F form a basis that decouples the system into modes. In particular, the eigenvectors can be divided into two sets. The first set spans the stable eigenspace while the second set spans the unstable eigenspace.

The eigenvectors of the Jacobian matrix of G are candidate basis vectors. Since the Jacobian matrix is a function of the phase vector, so are its eigenvectors. In general, the eigenvector basis will not isolate and dynamically decouple the stable and unstable components of G along a trajectory. The eigenvector basis may also not vary smoothly along a trajectory. The modified sweep method has some robustness to the basis used, but the basis selection is still crucial to the success of the method and is a topic of ongoing research. For purposes of this paper, it will be assumed that an eigenvector basis is used, but various constant and piecewise constant bases have also been used effectively for particular problems.

Initialization of q_u

Two facts guide the choice of the initial value for q_u. First, Eq. (29) states that the limiting value of q_u must be zero. Second, if modal basis vectors were used, q_u would be identically zero. Therefore, q_u identically zero seems a reasonable initialization. However, experience shows that using an initial guess of $q_u(t) \equiv 0$ is not always adequate for the iterative method (described next) to converge.

Forward Sweep

The initial condition for the adjoint, $\lambda(0)$, is determined from the algebraic equation $W_u(p_0) \cdot G(p_0) = q_u(0)$. A successive approximation approach can be used to solve for $\lambda(0)$. A starting value for $\lambda(0)$ is chosen. The Jacobian is then well-defined and its eigenvectors can be computed. The algebraic equation $W_u \cdot G(p_0) = q_u(0)$ can then be used to solve for a new value of $\lambda(0)$. Repeating this until convergence produces the initial adjoint $\lambda(0) = \lambda_0$. Using this value along with the initial state $x(0) = x_0$, an initial value q_{s0} can be computed from $q_{s0} = W_s \cdot G(p_0)$. Using $q_u(\cdot)$ as an input, the $3n$ first-order differential equations

$$\begin{aligned} \dot{p} &= V_u q_u + V_s q_s,\ p(0) = p_0 \\ \dot{q}_s &= \left[\dot{W}_s V + W_s J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_s(0) = q_{s0} \end{aligned} \tag{32}$$

are integrated forward to a time τ large enough to approximate the infinite horizon to sufficient accuracy.

Backward Sweep

A new iterant for q_u can be computed using the information from the forward sweep. At time $t = \tau$, the adjoint $\lambda(\tau)$ is known to be zero from the boundary condition. The value for $x(\tau)$ can be solved from $W_s(p(\tau)) \cdot G(p(\tau)) = q_s(\tau)$ where $q_s(\tau)$ is known from the forward sweep. Then $q_u(\tau)$ can be solved from $q_u(\tau) = W_u(p(\tau)) \cdot G(p(\tau))$. Using $q_s(t)$ from the forward sweep as an input, the $3n$ first-order differential equations

$$\begin{aligned} \dot{p} &= V_u q_u + V_s q_s,\ p(\tau) = \begin{bmatrix} x(\tau) \\ \lambda(\tau) \end{bmatrix} \\ \dot{q}_u &= \left[\dot{W}_u V + W_u J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_u(\tau) = 0 \end{aligned} \tag{33}$$

are integrated from $t = \tau$ to $t = 0$. The value of $q_u(t)$ generated from Eq. (33) is then used on the ensuing forward sweep.

Modified Sweep Method

The forward and backward sweeps are summarized as

(1) Initialization:

Choose $q_u(\cdot)$ (e.g., set $q_u(t) \equiv 0$).

(2) Generate Boundary Conditions at $t = 0$:

Solve $W_u(x_0, \lambda(0)) \cdot G(x_0, \lambda(0)) = q_u(0)$ for $\lambda(0) = \lambda_0$.
Compute $q_s(0) = q_{s0} = W_s(x_0, \lambda_0) \cdot G(x_0, \lambda_0)$.

(3) Forward Sweep: Given $q_u(t)$ from initialization or backward sweep

Integrate from $t = 0$ to $t = \tau$

$$\dot{p} = V_u q_u + V_s q_s,\ p(0) = p_0 = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$$
$$\dot{q}_s = \left[\dot{W}_s V + W_s J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_s(0) = q_{s0}$$

(4) Generate Boundary Conditions at $t = \tau$:

Solve $W_s(x(\tau), \lambda_f) \cdot G(x(\tau), \lambda_f) = q_s(\tau)$ for $x(\tau) = x_f$.
Compute $q_u(\tau) = q_{uf} = W_u \cdot G(x_f, \lambda_f)$.

(5) Backward Sweep: Given q_s from forward sweep

Integrate from $t = \tau$ to $t = 0$

$$\dot{p} = V_u q_u + V_s q_s,\ p(\tau) = \begin{bmatrix} x_f \\ \lambda_f \end{bmatrix}$$
$$\dot{q}_u = \left[\dot{W}_u V + W_u J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_u(\tau) = q_{uf}$$

(6) Convergence Check: if the difference between $p(\cdot)$ from the forward sweep and $p(\cdot)$ from the backward sweep is sufficiently small, then stop. Otherwise repeat (2)-(5).

Relation to Backward Sweep Method

Consider a basis of the form

$$V = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \qquad (34)$$

The dynamics of the corresponding phase rate coordinates along a trajectory of the Hamiltonian system in Eq. (6) are

$$\begin{aligned} \dot{q}_s &= (H_{\lambda x} + H_{\lambda\lambda} S) q_s + H_{\lambda\lambda} q_u \\ \dot{q}_u &= \Gamma(S) q_s - (H_{x\lambda} + S H_{\lambda\lambda}) q_u \end{aligned} \qquad (35)$$

where $\Gamma(S) = -\dot{S} - S H_{\lambda x} - S H_{\lambda\lambda} S - H_{xx} - H_{x\lambda} S$ is a Riccati differential equation. By choosing S to be a solution of $\Gamma(S) = 0$, q_u is decoupled from q_s. It can be shown that at any point along a trajectory of the Hamiltonian system of Eq. (6) lying on the stable manifold of a saddle-point equilibrium that the span of the n columns of $\begin{bmatrix} I \\ S \end{bmatrix}$ is the tangent space to the stable manifold. In the modified sweep method, the focal point is the phase rate vector G. Along a trajectory on the stable manifold, G lies in the tangent space to the stable manifold. Thus, if G is expressed in the basis just introduced, the q_u component is zero at any point $p = (x, \lambda)$ on the stable manifold.

Assume for the moment that the appropriate Riccati equation solution is known. The modified sweep method would then proceed by determining $\lambda(0) = \lambda_0$ such that $W_u \cdot G(x_0, \lambda_0) = -S H_\lambda - H_x = q_u(0) = 0$ which is the desired point on the stable manifold. The conventional sweep method on the other hand would use S to make small changes in $\lambda(0)$ from the previous iteration. Furthermore, the conventional sweep method requires integration of the Hamiltonian system with no control over the unstable behavior whereas the modified sweep method removes the contribution of the unstable behavior.

An approach to obtaining S for the class of problems considered in this paper is to compute its value at the final time from the algebraic Riccati equation or the local eigenvectors and initialize the modified sweep method at the final time. Other approaches need to be developed for other classes of problems.

Remarks on the Method

Convergence and Accuracy

Numerical Results. As with any numerical method for solving HTPBVP's, the modified sweep method is not guaranteed to converge from any arbitrary initialization. in fact, at the time, even local convergence has not been proved. However, it can be shown that *any* converged solution of the modified sweep method must satisfy the original differential equations given by Eq. (6). The method has been tested on example problems, but it is beyond the scope of this paper to show numerical results. In future reports on this method, specific examples will be included to illustrate the effectiveness of the method. In particular, current work is being done to show the effectiveness of the method for problems in supersonic aircraft trajectory optimization.

Choice of Basis Vectors. In the absence of numerical integration errors, the method will converge when modal basis vectors are used. This property is verified from the fact that the initial condition $\lambda(0)$ and the value $h_u(t) \equiv 0$ are both exact. In the case of non-modal basis vectors, no such guarantee can be made since the column-span of W_u will differ from that of B_u and will result in a wrong value of the adjoint from $W_u \cdot G = 0$. If the difference in the direction of B_u and W_u is sufficiently different, it may not be possible to produce a stable forward sweep. It may also turn out that for certain values of W_u, no solution may exist to $W_u \cdot G = 0$. As discussed above, the local eigenvectors of the Jacobian have been shown to work as basis vectors for several problems, but more work needs to be done to determine when the eigenvectors are reasonable basis vectors. This issue will be discussed in detail in a future report.

Conclusions

A new method for solving infinite horizon optimal control problems was introduced. This method is based on the computational singular perturbation methodology in which the state-adjoint vector is expressed in terms of a linear combination of known basis vectors and rate coordinates. Two categories of basis vectors were considered, modal and non-modal. When a modal basis is used, the stable rate coordinates are dynamically decoupled from the unstable rate coordinates. In this case the rate coordinates are called modal rate coordinates. Using a modal basis, the solution to the HTPBVP could be determined exactly.

A modified sweep method was presented for the case when modal basis vectors are not available. In this case the rate coordinates do not correspond to stable and unstable motion; their motions are coupled. The modified sweep method computes the coupling between the stable and unstable motion. In the modified sweep method, a set of non-modal basis vectors is computed and an initial guess is taken for the approximate unstable rate coordinates. On the forward sweep, a values for the state, adjoint, and the approximate stable rate coordinates are generated. The value for the approximate stable rate coordinates is then used on the backward sweep to compute a new value for the approximate unstable rate coordinates. Converged solutions to the modified sweep method can be shown to satisfy the original differential equations exactly.

Acknowledgments

The authors would like to thank Professor Sau -Hai Lam for his many helpful discussions. In addition, the authors would like to thank Andrew Tron for providing insight into many of the implementation issues. Finally, the authors would like to thank the National Science Foundation for supporting this research.

References

[1] Bryson, A. E., Desai, M. N., and Hoffman, W. C., "Energy-state approximation in performance optimization of supersonic aircraft," *Journal of Aircraft*, Vol. 6, No. 6, pp. 481–488, 1969.

[2] Kelley, H. J., and Edelbaum, T. N., "Energy climbs, energy turns, and asymptotic expansions," *Journal of Aircraft*, Vol. 7 No. 1, pp. 93–95, 1970.

[3] Kelley, H. J., "Reduced order modeling in aircraft mission analysis," *Journal of Aircraft*, Vol. 9, No. 2, pp. 349–350, 1971.

[4] Ardema, M. D., "Solution of the minimum time-to-climb problem by matched asymptotic expansions," *AIAA Journal*, Vol. 14, No. 7, pp. 843–850, 1976.

[5] Calise, A. J., "Singular perturbation methods for variational problems in aircraft flight," *IEEE Transactions on Automatic Control*, Vol. AC-21, No. 3, pp.345–353, 1976.

[6] Calise, A. J., "A new boundary layer matching procedure for singularly perturbed systems," *IEEE Transactions on Automatic Control*, Vol. AC-23, No. 3, pp. 434–438, 1978.

[7] Calise, A. J., "A singular perturbation analysis of optimal aerodynamic and thrust magnitude control," *IEEE Transactions on Automatic Control*, Vol. AC-24, No. 5, pp. 720–730, 1979.

[8] Khalil, H. K., Kokotovic, P. V., and O' Reilly, J., *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press, New York, 1986.

[9] Lam, S. H., and Goussis, D. A., "Basic theory and demonstrations of computational singular perturbation for stiff equations," *12th Annual IMACS World Congress of Scientific Computation*, Paris, France, July 18-22, 1998; IMACS Transactions of Scientific Computing, Numerical Methods and Applied Mathematics, C. Brezinski, Ed., J. C. Baltzer Scientific Publishing Co., pp. 487–492, 1988.

[10] Lam, S. H., and Goussis, D. A., "Conventional asymptotics and computational singular perturbation for simplified kinetics modeling," In *Reduced Mechanisms and Asymptotic Approximations for Methane-Air Flames*, Chapter 10: "Conventional Asymptotics and Computational Singular Perturbation for Simplified Chemical Kinetics Modeling," Lecture Notes in Physics, 284, M. Smooke, Ed., Springer-Verlag, 1991.

[11] Freedman, M. I., and Kaplan, J. L., "Singular perturbations of two-point boundary value problems arising in optimal control," *SIAM Journal on Control and Optimization*, Vol. 14, No. 2, pp. 189–215, 1976.

[12] Lichtenberg, A. J., and Lieberman, M. A., *Regular and Stochastic Motion*, Springer-Verlag, New York, 1983.

[13] Bryson, A. E., and Ho, Y-C, *Applied Optimal Control: Optimization, Estimation, and Control.* Hemisphere Publishing, New York, 1975.

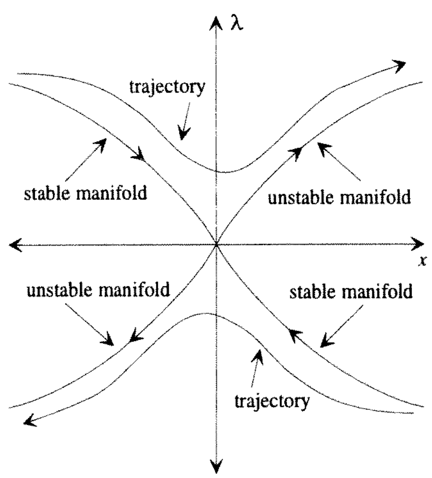

Figure 1: Depiction of geometric structure of Eq. (6) for the two-dimensional case.