A New Invariance Property of Lyapunov Characteristic Directions

S. Bharadwaj and K.D. Mease
Mechanical and Aerospace Engineering
University of California, Irvine, California, 92697-3975
Email: sanjay@eng.uci.edu, kmease@uci.edu

Abstract

Lyapunov exponents and direction fields are used to characterize the time-scales and geometry of general linear time-varying (LTV) systems of differential equations. Lyapunov exponents are already known to correctly characterize the time-scales present in a general LTV system; they reduce to real parts of eigenvalues when computed for linear time-invariant(LTI) systems and real parts of Floquet exponents when computed for periodic LTV systems. Here, we bring to light new invariance properties of Lyapunov direction fields to show that they are analogous to the Schur vectors of an LTI system and reduce to the Schur vectors when computed for LTI systems. We also show that the Lyapunov direction field corresponding to the smallest Lyapunov exponent when computed for an LTI system (with real distinct eigenvalues) reduces to the eigenvector corresponding to the smallest eigenvalue and when computed for a periodic LTV system (with real distinct Floquet exponents), reduces to the Floquet direction field corresponding to the smallest Floquet exponent.

1 Introduction

In the effort towards understanding the geometric structure and qualitative behavior of the state-space flow of smooth finite-dimensional nonlinear dynamical systems of the form

$$\dot{x} = f(x) \; , \; x \in \mathbb{R}^n \tag{1}$$

where $f(\cdot) : \mathbb{R}^n \to T\mathbb{R}^n$ is a smooth vector field, a commonly adopted first step is to study the linearized dynamics about a reference orbit $\gamma(\cdot)$ of Eq.(1), i.e., $\dot{\gamma} = f(\gamma(t))$. Linear time-varying (LTV) systems of interest to us typically arise from such a linearization about a reference orbit which we assume to be smooth and bounded. The resulting LTV system is given by

$$\delta\dot{x} = \left(\frac{\partial f(\gamma(t))}{\partial x}\right) = A(t)\delta x \tag{2}$$

where $\delta x \in T_x\mathbb{R}^n$ and where $A(t) \in \mathbb{R}^{n\times n}$ is assumed to have bounded entries. Recall that the transition matrix $\Phi(t,\tau)$ associated with the LTV system is given by

$$\frac{\partial \Phi}{\partial t} = A(t)\Phi \; , \; \Phi(\tau,\tau) = I_n \tag{3}$$

where I_n is the identity matrix of order n. Unlike LTI systems, no closed form expression exists in general relating the transition matrix Φ of an LTV system to $A(t)$. Also note that $\delta x(t) = \Phi(t,\tau)\delta x(\tau)$ for all $t,\tau \in \mathbb{R}$.

The LTV system determines how tangent vectors are mapped between the different tangent spaces $T_{\gamma(t)}\mathbb{R}^n$ along the reference trajectory. Knowledge of the directions of tangent vectors that grow/decay at extremal rates when mapped by the flow determined by the LTV system and their corresponding rates of expansion/contraction yields information about convergence/divergence of state trajectories neighboring $\gamma(\cdot)$ and their corresponding rates. A complete spectral characterization of the LTV dynamics aims at uncovering the different asymptotic rates of growth/decay of solutions of the LTV system along with their corresponding directions. Geometrically speaking, we seek linearly independent time-varying direction fields along $\gamma(\cdot)$ that identify the different characteristic directions on each tangent space $T_{\gamma(t)}\mathbb{R}^n$ along the reference trajectory.

When the reference trajectory is an isolated equilibrium point of Eq.(1), i.e., $\gamma(t) = \gamma_0$ for all $t \in \mathbb{R}$, the resulting LTV system becomes linear time-invariant (LTI) and is determined by $A(t) = A =$ constant for all $t \in \mathbb{R}$. The spectral structure of an LTI system, $\delta\dot{x} = A\delta x$, is completely characterized by the eigenvalues and eigenvectors of the matrix A. When the reference trajectory is an isolated periodic orbit of Eq.(1), i.e, $\gamma(t + kT) = \gamma(t)$ for all $t \in \mathbb{R}, k \in \mathbb{Z}$ where $T \in \mathbb{R}_+$ is the minimal period, the spectral structure of the resulting periodic LTV system is determined by Floquet theory [3]. A vast body of literature [1, 8, 10, 11, 12] exists on efforts to characterize the spectral structure of general LTV systems which arise when the reference trajectory is not necessarily either an equilibrium point or a periodic orbit, beginning with the pioneering work of Lyapunov [7] in his 1892 thesis. Lyapunov introduced the notion characteristic exponents to characterize solutions of LTV dynamics with extremal evolution rates which have since received a lot of attention from theoretical and computational perspectives [1, 2]. The same cannot be said of the associated Lyapunov direction fields. By investigating the invariance properties of Lyapunov direction fields we show that in addition to identifying directions of extremal growth/decay of solutions, they span invariant distributions (see [6] for a description of distributions) analogous to the invariant subspaces spanned by Schur vectors of LTI dynamics.

1.1 Lyapunov Exponents

The theory of Lyapunov exponents is described in well-known texts [5, 9]. For regular LTV systems [1, 2, 9], the Lyapunov exponents are well-defined limits on tangent subbundles of $\gamma(\cdot)$ on which the evolution rates are extremal. They are given by [5]

$$\begin{aligned} \mu_i[\delta x_i] &= \lim_{t\to\infty}\left(\frac{1}{t}\right)\ln\left(\|\delta x_i(t)\|\right) \\ \Rightarrow \quad \mu_i[\delta x_i] &= \lim_{t\to\infty}\left(\frac{1}{t-\tau}\right)\ln\left(\|\Phi(t,\tau)\delta x_i(\tau)\|\right) \end{aligned}$$

where $\delta x_i(t)$, $1 \leq i \leq n$ constitute a set of normal basis solutions [5] of Eq.(2). In the sequel, we assume that LTV systems of interest possess the property of regularity.

It is important to note that Lyapunov exponents are constants over the entire trajectory $\gamma(\cdot)$ and are independent of the starting point $\gamma(\tau)$. Also, the exponents are known to be invariant under Lyapunov transformations [5]. Lyapunov exponents computed for an LTI system yield the real parts of the eigenvalues and when computed for a periodic LTV system yield the real parts of the Floquet exponents (see [5],Thm. 63.4).

1.2 Lyapunov Directions

The Lyapunov directions at any given time along the reference trajectory $\gamma(\cdot)$ point in the direction of extremal average rates of growth/decay. So, they can be determined at any initial time τ by vectors $\delta x_i(\tau)$ which extremize $\|\Phi(t,\tau)\delta x_i(\tau)\|$ as $t \to \infty$. The solutions $\delta x_i(\tau)$ to this extremization problem is given by the eigenvectors of $\lim_{t\to\infty}(\Phi^T(t,\tau)\Phi(t,\tau))^{1/2t}$[2].

Greene and Kim [4] have shown that when the time-scales given by the Lyapunov exponents are distinct, the corresponding Lyapunov directions depend only on the position in state space given by $\gamma(\tau)$. The Lyapunov direction fields are everywhere orthogonal and determine directions at each point along the reference trajectory along which the average growth/decay rate is extremal. However, Lyapunov direction fields in general are not solutions of the LTV system. For example, when the Lyapunov directions are propagated forward from some time τ they do not in general remain orthogonal. In other words, not all of the Lyapunov direction fields are invariant under the linear flow of the LTV dynamics. In the following section we investigate the invariance properties of Lyapunov direction fields for regular LTV systems with distinct Lyapunov exponents.

2 Invariance Properties of Lyapunov Direction fields

The following lemma shows that when a particular Lyapunov direction is propagated forward by the linear flow, the time-evolved direction can develop components only along Lyapunov direction fields associated with smaller Lyapunov exponents.

Lemma 1 *Consider a regular LTV system of the form (2) which has the distinct ordered Lyapunov exponents $\mu_i, i = 1,\ldots,n$ where $\mu_1 > \ldots > \mu_n$ and corresponding time-varying directions $l_i(t), i = 1,\ldots,n$ for all $t \in \mathbb{R}$. Propagate these directions $l_i(\tau)$ forward from an initial time τ to $t, t > \tau$ using the transition matrix so that $m_i(t) = \Phi(t,\tau)l_i(\tau)$. Then*

$$\langle m_i(t), l_j(t)\rangle = \begin{cases} 0 & \text{if } i > j, \\ \text{nonzero} & \text{if } i \leq j \end{cases} \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and $1 \leq i, j \leq n$.

Proof: Recall that the Lyapunov exponents μ_i are computed by

$$\mu_i[l_i] = \lim_{t\to\infty}\left(\frac{1}{t-\tau}\right)\ln\left(\|\Phi(t,\tau)l_i(\tau)\|\right) \tag{5}$$

where the Lyapunov directions $l_i(t)$ are mutually orthogonal at each $t \in \mathbb{R}$ because they arise from the computation of eigenvectors of a symmetric matrix. Therefore

$$\langle l_i(\tau), l_j(\tau)\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

for all $1 \leq i, j \leq n$. However, when propagated forward from time τ to $t_1 > \tau$ using $\Phi(t_1,\tau)$, the propagated vectors $m_i(t_1) = \Phi(t_1,\tau)l_i(\tau)$ are in general no longer mutually orthogonal.

Let us assume that $\langle m_i(t_1), l_j(t_1)\rangle \neq 0$ for all $i > j$ and show that we arrive at a contradiction. Since vectors $l_j(t_1)$ are linearly independent (in fact orthogonal), we can express the vectors $m_i(t_1)$ as linear combinations of $l_j(t_1)$. For each i, let

$$m_i(t_1) = \sum_{j=1}^{n} c_{ij} l_j(t_1) \tag{6}$$

The above assumption then corresponds to $c_{ij} \neq 0$ for all $i > j$.

Eq. (6) implies that for each i,

$$\Phi(t, t_1) m_i(t_1) = \sum_{j=1}^{n} c_{ij} \Phi(t, t_1) l_j(t_1)$$

As t becomes large, the right hand side is dominated by the term corresponding to the vector $l_j, j = j^*$ for which $\Phi(t, t_1) l_{j^*}(t_1)$ is the largest and $c_{ij^*} \neq 0$. Since we have assumed the Lyapunov exponents (which determine the rates of change in size of vectors when propagated by the linear flow) to be arranged in descending order, the dominant term corresponds to when j^* is the smallest positive integer for which $c_{ij^*} \neq 0$.

Recall that the Lyapunov exponents μ_i are constants along the reference trajectory $\gamma(\cdot)$ and independent of the starting point $\gamma(\tau)$ on the reference trajectory. Compute the Lyapunov exponents from the initial directions $m_i(t_1)$ using

$$\nu_i = \lim_{t \to \infty} \left(\frac{1}{t - t_1} \right) \ln \left(\| \Phi(t, t_1) m_i(t_1) \| \right) \qquad (7)$$

In the light of the earlier argument, we can see that $\|\Phi(t, t_1) m_i(t_1)\| \to c_{ij^*} \|\Phi(t, t_1) l_{j^*}\|$ as $t \to \infty$ and that $\nu_i = \mu_{j^*}$.

This contradicts the fact that Lyapunov exponents are constants independent of the starting point on the reference trajectory unless $j^* = i$. The condition $j^* = i$ holds if and only if $c_{ij} = 0$ for all $i > j$. Since this discussion is valid for any time $t_1 > \tau$, we drop the subscript and conclude that

$$\langle m_i(t), l_j(t) \rangle = \begin{cases} 0 & \text{if } i > j, \\ \text{nonzero} & \text{if } i \leq j \end{cases}$$

for all times $t > \tau$. ∎

A direct consequence of the above lemma is the following theorem.

Theorem 1 *For any $1 \leq k \leq n$, the Lyapunov direction fields corresponding to the k smallest Lyapunov exponents of the LTV system (2) define an invariant distribution $\Delta(\gamma(t)) = span\{l_{n-k+1}(t), \ldots, l_n(t)\}$.*

Proof: Consider the Lyapunov directions $l_{n-k+1}(\tau), \ldots, l_n(\tau)$ at some initial time τ which correspond to the k smallest Lyapunov exponents. When propagated forward to time t, let $m_i(t) = \Phi(t, \tau) l_i(\tau)$ for all $n - k + 1 \leq i \leq n$. Lemma 1 implies that at time t, $span\{m_{n-k+1}(t), \ldots, m_n(t)\} = span\{l_{n-k+1}(t), \ldots, l_n(t)\}$. This is because, the directions $m_{n-k+1}(t), \ldots, m_n(t)$ cannot have components along any of the directions $l_1(t), \ldots, l_{n-k}(t)$. Consequently, if $v \in \Delta(\gamma(\tau))$ then $\Phi(t, \tau) v \in \Delta(\gamma(t))$ for all $t > \tau$. Therefore Δ is an invariant distribution. ∎

The above theorem and and lemma show that for regular LTV systems with distinct Lyapunov exponents the Lyapunov direction fields define a collection of invariant distributions $\Delta_k(\cdot) = span\{l_k, \ldots, l_n\}$ such that $dim(\Delta_k) = n - k + 1$ and $\Delta_1 \supset \ldots \supset \Delta_n$. When $\delta x(t) \in \Delta_i$ then $\|\delta x(t)\| \leq K e^{\mu_i t}$ where K is a positive constant. When these results are applied to an LTI system (typically resulting from linearizing nonlinear dynamics about an equilibrium point γ_0), the distributions Δ_i become subspaces W_i of $T_{\gamma_0}\mathbb{R}^n$. So the Lyapunov directions reduce to Schur vectors that span the respective nested invariant subspaces $W_1 \supset \ldots \supset W_n$. This establishes that the Lyapunov direction fields for an LTV system are time-varying analogs of Schur vectors for an LTI system.

Since a special case of the above theorem where $k = 1$ has many important consequences, we present it as a separate result.

Theorem 2 *The direction field $l_n(\cdot)$ corresponding to the smallest Lyapunov exponent μ_n is invariant when propagated using the transition matrix, i.e., $\Phi(t, \tau) l_n(\tau) = \sigma_n(t, \tau) l_n(t)$.*

Proof: For the LTV system (2) with Lyapunov exponents $\mu_1 > \ldots > \mu_n$, $l_n(t)$ is the direction field corresponding to the smallest Lyapunov exponent μ_n. Applying Lemma 1 to the direction $l_n(\tau)$ results in

$$\langle \Phi(t, \tau) l_n(\tau), l_j(t) \rangle = 0 \ , \ \forall \, j \neq n \qquad (8)$$

This implies that at time t, the direction $l_n(t)$ points in the same direction as $m_n(t) = \Phi(t, \tau) l_n(\tau)$, i.e,

$$\Phi(t, \tau) l_n(\tau) = \sigma_n(t, \tau) l_n(t) \qquad (9)$$

where $\sigma_n(t, \tau)$ is the scaling factor that determines that rate of growth of the size of $l_n(\tau)$. Hence we have shown that the Lyapunov direction field $l_n(\cdot)$ corresponding to the smallest Lyapunov exponent is invariant when propagated by the transition matrix. ∎

The following important corollaries and examples further justify the validity of the direction information provided by this Lyapunov direction field as an accurate generalization by showing that it is indeed consistent with the known results in the special cases of LTI systems and periodic LTV systems.

Corollary 1 *The Lyapunov direction corresponding to the smallest Lyapunov exponent, computed for an LTI* *system $\delta\dot{x} = A\delta x$ (which is assumed to have real and distinct eigenvalues ordered from largest to smallest), points in the same direction as the eigenvector direction corresponding to the smallest eigenvalue.*

Proof: As stated earlier, it is already known that the Lyapunov exponents of an LTI system are the same as (real parts of) the eigenvalues of A. Theorem 2 states that the Lyapunov direction l_n corresponding to the smallest Lyapunov exponent satisfies the condition of invariance when propagated using $\Phi(t,\tau) = \exp(A \cdot (t-\tau))$. Therefore $\Phi(t,\tau) l_n = \sigma_n(t-\tau) l_n$ where $\sigma_n = \exp(\mu_n \cdot (t-\tau))$, i.e., l_n is an eigenvector of Φ with eigenvalue σ_n which is the same as the eigenvector of A corresponding to the smallest eigenvalue. ∎

The following example shows that the Lyapunov directions computed for an LTI system reduce to the Schur vectors and in addition illustrates the above corollary.

Example 1 *Consider the LTI system $\delta\dot{x} = A\delta x$ where*

$$A = \begin{bmatrix} -6 & 5 \\ 4 & -5 \end{bmatrix} \tag{10}$$

The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = -10$ and the corresponding eigenvectors are $e_1 = [1;1]$ and $e_2 = [5;-4]$ respectively. The Schur vectors obtained by orthogonalizing the eigenvectors are $s_2 = e_2$ and $s_1 = [4;5]$.

The transition matrix of this LTI system is given by $\Phi(t,\tau) = \exp(A\cdot(t-\tau)) = (1/9)[\Phi_{ij}]$ where

$$\begin{aligned} \Phi_{11} &= 4e^{-(t-\tau)} + 5e^{-10(t-\tau)} \\ \Phi_{12} &= -5e^{-10(t-\tau)} + 5e^{-(t-\tau)} \\ \Phi_{21} &= -4e^{-10(t-\tau)} + 4e^{-(t-\tau)} \\ \Phi_{22} &= 5e^{-(t-\tau)} + 4e^{-10(t-\tau)} \end{aligned}$$

The eigenvalues of $\Phi^T(t,\tau)\Phi(t,\tau)$ are

$$\sigma_{1,2}(t,\tau) = (e^{-2(t-\tau)})(w \pm \sqrt{(w^2 - 6561e^{-18(t-\tau)})})$$

where $w = 41e^{-18(t-\tau)} - e^{-9(t-\tau)} + 41$. The Lyapunov exponents are given by

$$\mu_{1,2} = \lim_{t\to\infty} \frac{1}{2(t-\tau)} \ln(\sigma_{1,2}) \tag{11}$$

As $t \to \infty$, $\sigma_1(t,\tau) \to a\exp(-2(t-\tau))$ and $\sigma_2(t,\tau) \to b\exp(-20(t-\tau))$ where a and b are some constants whose true value is unimportant for our purpose. Therefore we find the Lyapunov exponents $\mu_1 = -1$ and $\mu_2 = -10$ which are the same as the eigenvalues λ_1 and λ_2 of A. The corresponding Lyapunov directions are given by the eigenvectors of $\Phi^T(t,\tau)\Phi(t,\tau)$ as $t \to \infty$. They are

$$l_{1,2} = \lim_{t\to\infty} \begin{bmatrix} -9e^{-9(t-\tau)} + 9 \mp (\sqrt{(w^2 - 6561e^{-18(t-\tau)})}) \\ 41e^{-18(t-\tau)} - e^{-9(t-\tau)} - 40 \end{bmatrix}$$

Therefore, $l_2 = [50;-40]$ and $l_1 = [-32;-40]$ which point in the same directions as the Schur vectors s_2 and s_1 respectively. Also note that l_2 points in the same direction as e_2.

We shall now show through the following corollary that the result of Theorem 2 is also consistent with the results of Floquet theory when applied to periodic LTV systems.

Corollary 2 *The Lyapunov direction field corresponding to the smallest Lyapunov exponent, computed for a periodic LTV system $\delta\dot{x} = A(t)\delta x$ with $A(t+T) = A(t)$ (which is assumed to have real and distinct Floquet exponents ordered from largest to smallest) points in the same direction as the Floquet direction field corresponding to the smallest Floquet exponent.*

Proof: As stated earlier, it is already known that the Lyapunov exponents of a periodic LTV system are the same as (real parts of) the Floquet exponents. Theorem 2 states that the Lyapunov direction field $l_n(\cdot)$ corresponding to the smallest Lyapunov exponent satisfies the invariance condition. Therefore

$$\Phi(t,\tau) l_n(\tau) = \sigma_n(t,\tau) l_n(t) \tag{12}$$

where $\sigma_n = \exp(\mu_n \cdot (t-\tau))$. We know that the Lyapunov direction fields are dependent only on the position in state space along the reference trajectory $\gamma(\cdot)$. In this case, $\gamma(\cdot)$ is periodic, i.e., $\gamma(\tau+T) = \gamma(\tau)$. Consequently, $l_n(\tau+T) = l_n(\tau)$, i.e., $l_n(\cdot)$ is periodic. Together with the invariance condition, we get

$$\Phi(\tau+T,\tau) l_n(\tau) = \exp(\mu_n T) l_n(\tau) \tag{13}$$

This implies that $l_n(\tau)$ is an eigenvector of $\Phi(\tau+T,\tau)$ corresponding to Floquet exponent $\lambda_n = \mu_n$. This shows that $l_n(\cdot)$ is a periodic direction field pointing in the same direction as the Floquet direction field corresponding to the smallest Floquet exponent. ∎

We shall illustrate this corollary with the following example.

Example 2 *Consider the periodic LTV system $\delta\dot{x} =$*

$A(t)\delta x$ where $A(t) = [A_{ij}(t)]$ so that

$$\begin{aligned}
A_{11}(t) &= -\frac{1}{2}\cos 2t + \frac{9}{2}\sin 2t - \frac{11}{2} \\
A_{12}(t) &= \frac{1}{2}\sin 2t + \frac{9}{2}\cos 2t + \frac{3}{2} \\
A_{21}(t) &= \frac{1}{2}\sin 2t + \frac{9}{2}\cos 2t - \frac{3}{2} \\
A_{22}(t) &= -\frac{9}{2}\sin 2t + \frac{1}{2}\cos 2t - \frac{11}{2}
\end{aligned}$$

with period $T = \pi$. This system was constructed from the LTI system in example 1 represented here as $\delta\dot{\mathsf{x}} = B\delta\mathsf{x}$ with a periodic transformation $\delta\mathsf{x} = S(t)\delta x$ where

$$B = \begin{bmatrix} -6 & 5 \\ 4 & -5 \end{bmatrix}, \; S(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \quad (14)$$

so that $A(t) = S^{-1}(t)BS(t) - S^{-1}(t)\dot{S}(t)$. The Floquet exponents turn out to be $\lambda_1 = -1$ and $\lambda_2 = -10$. The corresponding periodic Floquet direction fields are given by

$$e_1(t) = \begin{bmatrix} \cos t + \sin t \\ -\sin t + \cos t \end{bmatrix}, \; e_2(t) = \begin{bmatrix} 5\cos t - 4\sin t \\ -5\sin t - 4\cos t \end{bmatrix}$$

The transition matrix for this periodic LTV system is given by

$$\Omega(t,\tau) = S^T(t)\Phi(t,\tau)S(\tau) \quad (15)$$

where $\Phi(t,\tau)$ is the transition matrix in example 1. The Lyapunov exponents and associated directions are computed from the eigenvalues and eigenvectors of $\Omega^T(t,\tau)\Omega(t,\tau)$ as $t \to \infty$. Note that

$$\Omega^T(t,\tau)\Omega(t,\tau) = S^T(\tau)\Phi^T(t,\tau)\Phi(t,\tau)S(\tau) \quad (16)$$

because $S(t)S^T(t) = I_n$. Consequently, the eigenvalues of $\Omega^T\Omega$ are the same as the eigenvalues of $\Phi^T\Phi$ and its eigenvectors are related to that of $\Phi^T\Phi$ by the transformation $S^T(\tau)$. Therefore, the Lyapunov exponents of the periodic LTV system turn out to be $\mu_1 = -1$ and $\mu_2 = -10$ which are the same as the Floquet exponents. The direction field corresponding to the smallest Lyapunov exponent μ_2 computed from the eigenvector of $\Omega^T(t,\tau)\Omega(t,\tau)$ as $t \to \infty$ is given by

$$l_2(\tau) = S^T(\tau)\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 5\cos\tau - 4\sin\tau \\ -5\sin\tau - 4\cos\tau \end{bmatrix} \quad (17)$$

which is the same as the Floquet direction field corresponding to the smallest Floquet exponent.

3 Conclusions

Lyapunov direction fields span invariant distributions on which the average asymptotic evolution rates of solutions of the LTV dynamics are extremal. They reduce to Schur vectors when computed for an LTI system thereby establishing that they are appropriate time-varying analogs of Schur vectors. In particular, the Lyapunov direction field corresponding to the smallest Lyapunov exponent is invariant under the linear flow and this direction field is consistent with the corresponding eigenvector when computed for LTI dynamics and with the corresponding Floquet direction field when computed for periodic LTV dynamics. Illuminating examples were presented to illustrate these results.

References

[1] F. Colonius and W. Kliemann, "The Lyapunov Spectrum of Families of Time-Varying Matrices," *Transactions of The American Mathematical Society*, Vol. 348, No. 11, pp. 4389–4408, 1996.

[2] L. Dieci, R. D. Russell and E. S. Van Vleck, "On the Computation of Lyapunov Exponents for Continuous Dynamical Systems," *SIAM Journal of Numerical Analysis*, Vol. 34, No. 1, pp. 402–423, 1997.

[3] M. Farkas, *Periodic Motions*, Springer-Verlag, New York, 1994.

[4] J. M. Greene, and J.-S. Kim, "The Calculation of Lyapunov Spectra," *Physica D*, Vol. 24, pp. 213–225, 1987.

[5] W. Hahn, *Stability of Motion*, Springer-Verlag, New York, 1967.

[6] A. Isidori, "Nonlinear Control Systems", Second edition, *Springer*, 1994.

[7] A. M. Lyapunov, "The General Problem of Stability of Motion," *International Journal of Control*, Vol. 55, No. 3, pp. 531-773, 1992.

[8] R. J. Sacker and G. R. Sell, "A Spectral Theory for Linear Differential Systems," *Journal of Differential Equations*, Vol. 27, pp. 320-358, 1978.

[9] G. Sansone and R. Conti, *Nonlinear Differential Equations.* Pergamon Press, New York, 1964.

[10] W. E. Wiesel, "Optimal Pole Placement in Time-Dependent Linear Systems," *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 5, pp. 995–999, 1995.

[11] M.-Y. Wu, "A New Concept of Eigenvalues and Eigenvectors and its Applications," *IEEE Transactions on Automatic Control*, Vol. 25, No. 4, pp. 824–826, 1980.

[12] J. J. Zhu, "A Unified Spectral Theory for Linear Time-Varying Systems - Progress and Challenges," *Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans*, IEEE, New York, Vol. 3, pp. 2540–2546, 1995.